Lecture 5.5: *p*-groups

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Math 4120, Modern Algebra

Coming soon: the Sylow theorems

Definition

A *p-group* is a group whose order is a power of a prime p. A p-group that is a subgroup of a group G is a *p-subgroup* of G.

Notational convention

Throughout, G will be a group of order $|G| = p^n \cdot m$, with $p \nmid m$. That is, p^n is the *highest power* of p dividing $|G|$.

There are three Sylow theorems, and loosely speaking, they describe the following about a group's p-subgroups:

1. **Existence**: In every group, p-subgroups of all possible sizes exist.
2. **Relationship**: All maximal p-subgroups are conjugate.
3. **Number**: There are strong restrictions on the number of p-subgroups a group can have.

Together, these place strong restrictions on the structure of a group G with a fixed order.

p-groups

Before we introduce the Sylow theorems, we need to better understand p-groups.

Recall that a p-group is any group of order p^n. For example, C_1, C_4, V_4, D_4 and Q_4 are all 2-groups.

p-group Lemma

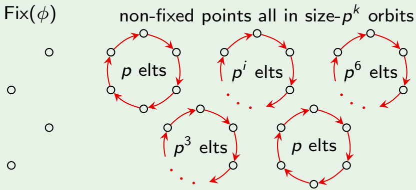
If a p-group G acts on a set S via $\phi \colon G \to \mathrm{Perm}(S)$, then

$$|\,\mathrm{Fix}(\phi)| \equiv_p |S|\,.$$

Proof (sketch)

Suppose $|G| = p^n$.

By the Orbit-Stabilizer theorem, the only possible orbit sizes are $1, p, p^2, \dots, p^n$.



Normalizer lemma, Part 1

If H is a p-subgroup of G, then

$$[N_G(H)\colon H] \equiv_p [G\colon H]\,.$$

Proof

Let $S = G/H = \{Hx \mid x \in G\}$. The group H acts on S by **right-multiplication**, via $\phi\colon H \to \mathrm{Perm}(S)$, where

$$\phi(h) = \text{the permutation sending each } Hx \text{ to } Hxh.$$

The fixed points of ϕ are the cosets Hx in the normalizer $N_G(H)$:

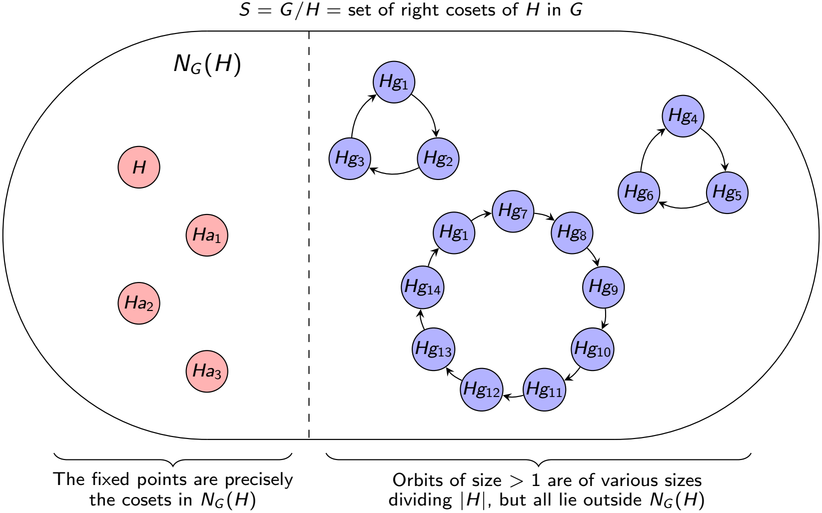
$$\begin{aligned} Hxh = Hx, \quad \forall h \in H \qquad &\Longleftrightarrow \qquad Hxhx^{-1} = H, \quad \forall h \in H \\ &\Longleftrightarrow \qquad xhx^{-1} \in H, \quad \forall h \in H \\ &\Longleftrightarrow \qquad x \in N_G(H)\,. \end{aligned}$$

Therefore, $|\,\mathrm{Fix}(\phi)| = [N_G(H)\colon H]$, and $|S| = [G : H]$. By our p-group Lemma,

$$|\,\mathrm{Fix}(\phi)| \equiv_p |S| \quad \Longrightarrow \quad [N_G(H)\colon H] \equiv_p [G\colon H]\,. \qquad \square$$

p-groups

Here is a picture of the action of the p-subgroup H on the set $S = G/H$, from the proof of the Normalizer Lemma.



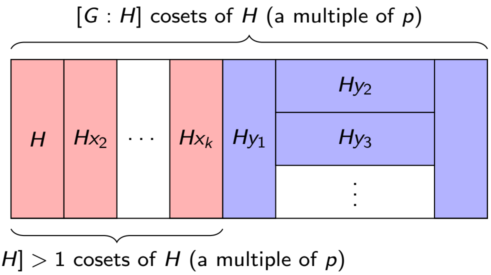
p-subgroups

The following result will be useful in proving the first Sylow theorem.

The Normalizer lemma, Part 2

Suppose $|G| = p^n m$, and $H \leq G$ with $|H| = p^i < p^n$. Then $H \lneq N_G(H)$, and the index $[N_G(H) : H]$ is a multiple of p.

$H \lneq N_G(H) \leq G$



Conclusions:

- $H = N_G(H)$ is impossible!
- p^{i+1} divides $|N_G(H)|$.

Proof of the normalizer lemma

The Normalizer lemma, Part 2

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Proof

Since $H \lhd N_G(H)$, we can create the quotient map

$$q \colon N_G(H) \longrightarrow N_G(H)/H\,, \qquad\qquad q \colon g \longmapsto gH\,.$$

The size of the quotient group is $[N_G(H) \colon H]$, the number of cosets of H in $N_G(H)$.

By The Normalizer lemma Part 1, $[N_G(H) \colon H] \equiv_p [G \colon H]$. By Lagrange's theorem,

$$[N_G(H) \colon H] \equiv_p [G \colon H] = \frac{|G|}{|H|} = \frac{p^n m}{p^i} = p^{n-i} m \equiv_p 0\,.$$

Therefore, $[N_G(H) \colon H]$ is a multiple of p, so $N_G(H)$ must be strictly larger than H. $\square$