Lecture 5.7: Finite simple groups

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Overview

Definition

A group G is simple if its only normal subgroups are G and $\langle e \rangle$.

Since all Sylow p-subgroups are conjugate, the following result is straightforward:

Proposition (HW)

A Sylow p-subgroup is normal in G if and only if it is the unique Sylow p-subgroup (that is, if $n_p = 1$).

The Sylow theorems are very useful for establishing statements like:

There are no simple groups of order k (for some k).

To do this, we usually just need to show that $n_p = 1$ for some p dividing $|G|$.

Since we established $n_5 = 1$ for our running example of a group of size $|M| = 200 = 2^3 \cdot 5^2$, there are no simple groups of order 200.

An easy example

Tip

When trying to show that $n_p = 1$, it's usually more helpful to analyze the largest primes first.

Proposition

There are no simple groups of order 84.

Proof

Since $|G| = 84 = 2^2 \cdot 3 \cdot 7$, the Third Sylow Theorem tells us:

- n_7 divides $2^2 \cdot 3 = 12$ (so $n_7 \in \{1, 2, 3, 4, 6, 12\}$)
- $n_7 \equiv_7 1$.

The only possibility is that $n_7 = 1$, so the Sylow 7-subgroup must be normal. □

Observe why it is beneficial to use the largest prime first:

- n_3 divides $2^2 \cdot 7 = 28$ and $n_3 \equiv_3 1$. Thus $n_3 \in \{1, \cancel{2}, 4, 7, \cancel{14}, 28\}$.
- n_2 divides $3 \cdot 7 = 21$ and $n_2 \equiv_2 1$. Thus $n_2 \in \{1, 3, 7, 21\}$.

A harder example

Proposition

There are no simple groups of order 351.

Proof

Since $|G| = 351 = 3^3 \cdot 13$, the Third Sylow Theorem tells us:

- n_{13} divides $3^3 = 27$ (so $n_{13} \in \{1, 3, 9, 27\}$)
- $n_{13} \equiv_{13} 1$.

The only possibilies are $n_{13} = 1$ or 27.

A Sylow 13-subgroup P has order 13, and a Sylow 3-subgroup Q has order $3^3 = 27$. Therefore, $P \cap Q = \{e\}$.

Suppose $n_{13} = 27$. Every Sylow 13-subgroup contains 12 non-identity elements, and so G must contain $27 \cdot 12 = 324$ elements of order 13.

This leaves $351 - 324 = 27$ elements in G not of order 13. Thus, G contains only one Sylow 3-subgroup (i.e., $n_3 = 1$) and so G cannot be simple. □

The hardest example

Proposition

If $H \lneq G$ and $|G|$ does not divide $[G:H]!$, then G cannot be simple.

Proof

Let G act on the **right cosets** of H (i.e., $S = G/H$) by **right-multiplication**:

$$\phi \colon G \longrightarrow \mathrm{Perm}(S) \cong S_n\,, \qquad \phi(g) = \text{the permutation that sends each } Hx \text{ to } Hxg.$$

Recall that the kernel of ϕ is the intersection of all conjugate subgroups of H:

$$\mathrm{Ker}\,\phi = \bigcap_{x \in G} x^{-1} H x.$$

Notice that $\langle e \rangle \leq \mathrm{Ker}\,\phi \leq H \lneq G$, and $\mathrm{Ker}\,\phi \lhd G$.

If $\mathrm{Ker}\,\phi = \langle e \rangle$ then $\phi \colon G \hookrightarrow S_n$ is an embedding. But this is *impossible* because $|G|$ does not divide $|S_n| = [G:H]!$. □

Corollary

There are no simple groups of order 24.

Theorem (classification of finite simple groups)

Every finite simple group is isomorphic to one of the following groups:

- A cyclic group $\mathbb{Z}_p$, with p prime;
- An alternating group A_n, with $n \geq 5$;
- A Lie-type Chevalley group: $\mathrm{PSL}(n,q)$, $\mathrm{PSU}(n,q)$, $\mathrm{PsP}(2n,p)$, and $P\Omega^{\epsilon}(n,q)$;
- A Lie-type group (twisted Chevalley group or the Tits group): $D_4(q)$, $E_6(q)$, $E_7(q)$, $E_8(q)$, $F_4(q)$, ${}^2F_4(2^n)'$, $G_2(q)$, ${}^2G_2(3^n)$, ${}^2B(2^n)$;
- One of 26 exceptional "sporadic groups."

The two largest sporadic groups are the:

- "baby monster group" B, which has order

$$|B| = 2^{41} \cdot 3^{13} \cdot 5^{6} \cdot 7^{2} \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47 \approx 4.15 \times 10^{33};$$

- "monster group" M, which has order

$$|M| = 2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \approx 8.08 \times 10^{53}.$$

The proof of this classification theorem is spread across $\approx$ 15,000 pages in $\approx$ 500 journal articles by over 100 authors, published between 1955 and 2004.

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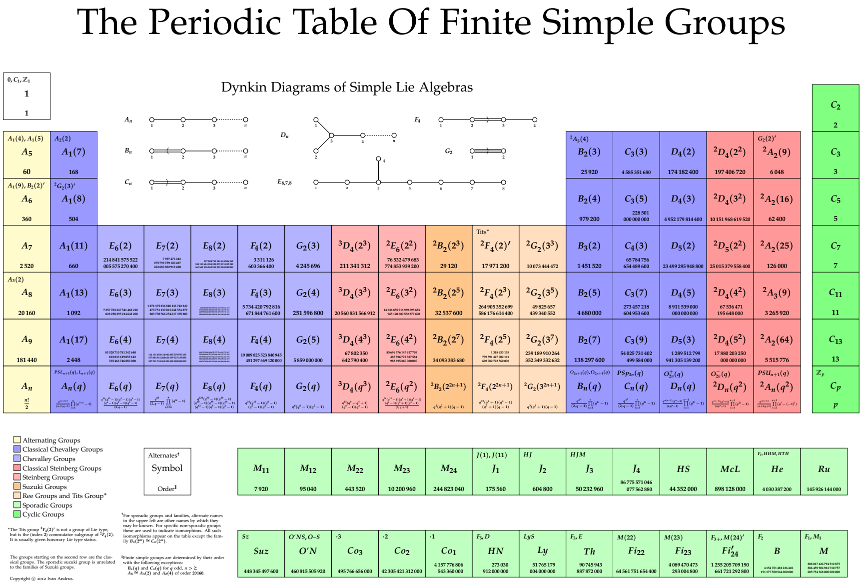

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