Lecture 4.1: Basic matrix algebra

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Math 2080, Differential Equations

Introduction

To analyze systems of differential equations, we need to heavily use matrix theory (linear algebra). We'll review this now, but will mostly focus on 2×2 matrices.

Adding and subtracting matrices

- Addition is "obvious": $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} =$
- Multiplication is trickier: $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} =$

Matrices and systems of equations

Application of matrices

Consider the following system of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

2×2 systems

Examples

1. Solve the system $\begin{cases} 3x_1 - x_2 = 8 \\ x_1 + 2x_2 = 5 \end{cases}$
2. Solve the system $\begin{cases} x_1 + 2x_2 = 1 \\ x_1 + 2x_2 = 5 \end{cases}$
3. Solve the system $\begin{cases} 2x_1 + 4x_2 = 10 \\ x_1 + 2x_2 = 5 \end{cases}$

Determinants

Example

Consider the system $\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$

The identity matrix and inverses

Definition

The 2×2 identity matrix is $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Solving $\mathbf{Ax} = \mathbf{b}$

The inverse of a 2×2 matrix

The inverse of the matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (when it exists), is

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

When $\mathbf{A}^{-1}$ exists, we say that $\mathbf{A}$ is invertible, or nonsingular. Otherwise, $\mathbf{A}$ is noninvertible, or singular.

A geometric way to view matrices

Observation

A 2×2 matrix is a linear map from $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$:

$$\mathbf{A}\colon \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \qquad \mathbf{x} \longmapsto \mathbf{y} = \mathbf{A}\mathbf{x}$$