Lecture 4.2: Eigenvalues and eigenvectors

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Preliminaries

Definition

Let $\mathbf{A}$ be a square matrix. A vector $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ if $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ for some constant λ, which is called an eigenvalue.

The trace of a matrix

Definition

The trace of $\mathbf{A}$ is the sum of the entries along the "main diagonal", and is denoted $\operatorname{tr}\mathbf{A}$. If $\mathbf{A}$ is 2×2:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad \operatorname{tr}\mathbf{A} = a_{11} + a_{22}\,.$$

Useful facts

1. For any square matrix, the trace is equal to the sum of the eigenvalues.
2. The characteristic polynomial of a 2×2 matrix $\mathbf{A}$ is

$$p(\lambda) = \det(A - \lambda I) = \lambda^2 - (\operatorname{tr}\mathbf{A})\lambda + \det\mathbf{A}\,.$$

Case 1: real distinct eigenvalues

Example 1

Find the eigenvalues and eigenvectors of $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$.

Case 2: complex eigenvalues

Example 2

Find the eigenvalues and eigenvectors of $\mathbf{A} = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix}$.

Case 3a: repeated eigenvalues, one eigenvector

Example 3a

Find the eigenvalues and eigenvectors of $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$.

Case 3b: repeated eigenvalues, two eigenvectors

Example 3b

Find the eigenvalues and eigenvectors of $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.