Lecture 4.3: Mixing problems with two tanks

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Motivation

Example

Suppose Tank A has 30 gallons of water containing 55 ounces of dissolved salt, and Tank B has 20 gallons of water containing 26 ounces of dissolved salt. Moreover:

- Water with salt concentration 1 oz/gal flows into Tank A at a rate of 1.5 gal/min.
- Water with salt concentration 3 oz/gal flows into Tank B at a rate of 1 gal/min.
- Water flows from Tank A to Tank B at a rate of 3 gal/min.
- Water flows from Tank B to Tank A at a rate of 1.5 gal/min.
- Water drains from Tank B at a rate of 2.5 gal/min.

Find equations $x_1(t)$ and $x_2(t)$ governing the amount of salt in Tanks A and B.

The steady-state solution

Example (cont.)

We derived an initial value problem $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$, $\mathbf{x}(0) = \mathbf{x}_0$:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1.5 \\ 3 \end{bmatrix}, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 55 \\ 26 \end{bmatrix}.$$

Since this is inhomogeneous, the general solution will have the form $\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t)$.

Changing variables

Example (cont.)

We derived an initial value problem $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$, $\mathbf{x}(0) = \mathbf{x}_0$:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1.5 \\ 3 \end{bmatrix}, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 55 \\ 26 \end{bmatrix}.$$

which has steady-state solution $\mathbf{x}_{ss} = \begin{bmatrix} 42 \\ 36 \end{bmatrix}$.

Summary

To solve the inhomogeneous initial value problem $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$, $\mathbf{x}(0) = \mathbf{x}_0$:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1.5 \\ 3 \end{bmatrix}, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 55 \\ 26 \end{bmatrix}$$

we take the following steps:

1. Find the steady-state solution, $\mathbf{x}_{ss} = \begin{bmatrix} 42 \\ 36 \end{bmatrix}$.
2. Change variables to convert the system into a homogeneous system, $\mathbf{y}' = \mathbf{A}\mathbf{y}$, $\mathbf{y}(0) = \mathbf{y}_0$:

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \qquad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 13 \\ -10 \end{bmatrix}$$

3. In the next lecture we will learn how to find the solution $\mathbf{y}(t)$ to this homogeneous system. Our general solution will be

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t) = \mathbf{y}(t) + \mathbf{x}_{ss}\,.$$