Lecture 4.4: Solving 2×2 homogeneous systems of ODEs

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A motivating example

Example 0

Consider the following homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}, \;\; \mathbf{x}(0) = \mathbf{x}_0$:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

The role of eigenvectors and eigenvalues

Theorem

If $\mathbf{Av} = \lambda \mathbf{v}$, then $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ solves $\mathbf{x}' = \mathbf{Ax}$.

Solving $\mathbf{x}' = \mathbf{A}\mathbf{x}$: The big ideas

Theorem

The general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is a two-parameter family $\mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t)$.

Therefore, if $\mathbf{A}$ has distinct eigenvalues $\lambda_1 \neq \lambda_2$ and eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, then the general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 \,.$$

Not surprisingly, the initial condition $\mathbf{x}(0) = \mathbf{x}_0 = \begin{bmatrix} a \\ b \end{bmatrix}$ determines a unique particular solution.

Case 1a: distinct, real (negative) eigenvalues

Example 1a (revisited)

Consider the following homogeneous system $\mathbf{y}' = \mathbf{A}\mathbf{y}$, $\mathbf{y}(0) = \mathbf{y}_0$:

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -0.1 & 0.075 \\ 0.1 & 0.2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \qquad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 13 \\ -10 \end{bmatrix}.$$

It is eaily verified that the eigenvalues and eigenvectors are

$$\lambda_1 = -0.25, \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}; \qquad\qquad \lambda_2 = -0.05, \qquad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Phase portraits

Example 1a (cont.)

The general solution to our system $\mathbf{y}' = \mathbf{A}\mathbf{y}$, $\mathbf{y}(0) = \mathbf{y}_0$ is the two-parameter family

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = C_1 e^{-0.25t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 e^{-0.05t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} .$$

The phase portrait of this system is the plot $y_2(t)$ vs. $y_1(t)$.