Lecture 4.6: Phase portraits with complex eigenvalues

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Complex-valued solutions

Lemma

Suppose $\mathbf{x}_1(t) = \mathbf{u}(t) + i\mathbf{w}(t)$ solves $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Then so do $\mathbf{u}(t)$ and $\mathbf{w}(t)$.

Re-writing the general solution

Example 2a

Consider the following homogeneous system $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

It is easily verified that the eigenvalues and eigenvectors of $\mathbf{A}$ are

$$\lambda_1 = -\frac{1}{2} + i, \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}; \qquad\qquad \lambda_2 = -\frac{1}{2} - i, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

Thus, the general solution is $\mathbf{x}(t) = C_1 e^{(-\frac{1}{2}+i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} + C_2 e^{(-\frac{1}{2}-i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix}$.

Phase portraits

Example 2a (cont.)

Let's plot the general solution $\mathbf{x}(t) = C_1 e^{-t/2} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + C_2 e^{-t/2} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$.

Purely imaginary eigenvalues

Example 2a

Consider the following initial value problem $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{5}{4} \\ 2 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$.
It is easily verified that the eigenvalues and eigenvectors of $\mathbf{A}$ are

$$\lambda_1 = \frac{3}{2} i, \qquad \mathbf{v}_1 = \begin{bmatrix} 5 \\ 2 - 6i \end{bmatrix}; \qquad\qquad \lambda_2 = -\frac{3}{2} i, \qquad \mathbf{v}_2 = \begin{bmatrix} 5 \\ 2 + 6i \end{bmatrix}.$$

Thus, the general solution is $\mathbf{x}(t) = C_1 e^{\frac{3}{2} it} \begin{bmatrix} 5 \\ 2 - 6i \end{bmatrix} + C_2 e^{-\frac{3}{2} it} \begin{bmatrix} 5 \\ 2 + 6i \end{bmatrix}$.

Concluding remarks

Summary

Suppose we wish to solve $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\lambda_{1,2} = a \pm bi$, $b \neq 0$.

We have two solutions: $\mathbf{x}_1(t) = e^{(a+bi)t}\mathbf{v}_1$ and $\mathbf{x}_2(t) = e^{(a-bi)t}\mathbf{v}_2$.

Take one of these (say $\mathbf{x}_1(t)$), and use Euler's formula to write it as $\mathbf{x}_1(t) = \mathbf{u}(t) + i\mathbf{w}(t)$.

The general solution is $\mathbf{x}(t) = C_1\mathbf{u}(t) + C_2\mathbf{w}(t)$.

The phase portrait will have ellipses, that are

- spiraling inward if $a < 0$;
- spiraling outward if $a > 0$;
- stable if $a = 0$.