Lecture 4.9: Variation of parameters for systems

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Math 2080, Differential Equations

Variation of parameters for non-systems

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It turns out that

$$v_1(t) = \int \frac{-y_2(t)f(t)\,dt}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}, \qquad v_2(t) = \int \frac{y_1(t)f(t)\,dt}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}.$$

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These methods *always work*, assuming that you can find $y_h(t)$, and evaluate the integrals.

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A specific example

Example 1

Solve the initial value problem $\boldsymbol{x}' = \begin{bmatrix} 1 & -4 \\ 2 & -5 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 10\cos t \\ 2e^{-t} \end{bmatrix}, \qquad \boldsymbol{x}(0) = \begin{bmatrix} 10 \\ 4 \end{bmatrix}.$

A general example

Example 2

Solve $y'' + p(t)y' + q(t)y = f(t)$ by turning it into a 2×2 system first.

Summary

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(i) $y_p(t) = v(t)y_1(t)$ (1st order)
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- This method also works if $A(t)$ is non-constant, assuming that we can actually find $\boldsymbol{x}_h(t) = C_1\boldsymbol{x}_1(t) + C_2\boldsymbol{x}_2(t)$.
- Such a solution is only defined where the Wronskian

$$W[\boldsymbol{x}_1(t), \boldsymbol{x}_2(t)] := \det \begin{bmatrix} x_{11}(t) & x_{21}(t) \\ x_{12}(t) & x_{22}(t) \end{bmatrix}, \quad \textit{or} \quad W[y_1(t), y_2(t)] := \det \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}$$

is non-zero.