Lecture 7.1: The heat equation

Matthew Macauley

Department of Mathematical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

Math 2080, Differential Equations

Partial differential equations

Definition

Let $u(x,t)$ be a 2-variable function. A partial differential equation (PDE) is an equation involving u, x, t, and the partial deriatives of u.

PDEs vs. ODEs

ODEs have a unifying theory of existence and uniqueness of solutions.

PDEs have no such theory.

PDEs arise from physical phenomena and modeling.

A simple one-dimensional PDE

The heat equation

Consider a bar of length L that is insulated along its interior. The heat equation is the PDE

$$\rho(x)\sigma(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(\kappa(x)\frac{\partial u}{\partial x}\Big), \qquad \text{where}$$

$$\begin{aligned} u(x,t) &= \text{ temperature of the bar at position } x \text{ and time } t \\ \rho(x) &= \text{ density of the bar at position } x \\ \sigma(x) &= \text{ specific heat at position } x \\ \kappa(x) &= \text{ thermal conductivity at position } x \end{aligned}$$

We'll assume that the bar is "uniform" (i.e., ρ, σ, and κ are constant). In this case, the heat equation becomes

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad c^2 = \frac{\kappa}{\rho\sigma}.$$

Dirichlet boundary conditions (1st type)

Example 1a

Solve the following initial/boundary value problem:

$$u_t = c^2 u_{xx}, \qquad u(0,t) = u(\pi,t) = 0, \qquad u(x,0) = x(\pi - x)\,.$$

Dirichlet boundary conditions (1st type)

Example 1a (cont.)

The general solution to the following BVP for the heat equation:

$$u_t = c^2 u_{xx}, \qquad u(0,t) = u(\pi,t) = 0, \qquad u(x,0) = x(\pi - x)\,.$$

is $u(x,t) = \sum_{n=1}^{\infty} b_n \sin nx \, e^{-c^2 n^2 t}$. Now, we'll solve the remaining IVP.