Lecture 7.6: Laplace's equation

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Math 2080, Differential Equations

Overview

Definition

A function $u(x_1, \dots, x_n)$ is harmonic if any of the following conditions hold:

- $\nabla^2 u := \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0$,
- u is a steady-state solution to the heat equation $u_t = \nabla^2 u$ (for some BCs),
- the graph of u is "as flat as possible"

The PDE $\nabla^2 u = 0$ is called Laplace's equation.

Solving Laplace's equation

Example 1a

Solve the following BVP for Laplace's equation:

$$u_{xx} + u_{yy} = 0, \qquad u(0,y) = u(x,0) = u(\pi,y) = 0, \quad u(x,\pi) = x(\pi - x)\,.$$

Solving Laplace's equation

Example 1b

Solve the following BVP for Laplace's equation:

$$u_{xx} + u_{yy} = 0, \qquad u(0,y) = u(x,0) = u(x,\pi) = 0, \quad u(\pi,y) = y(\pi - y)\,.$$

Solving Laplace's equation

Example 1c

Solve the following BVP for Laplace's equation:

$$u_{xx} + u_{yy} = 0, \qquad u(0, y) = u(x, 0) = 0, \quad u(x, \pi) = x(\pi - x), \quad u(\pi, y) = y(\pi - y)\,.$$