Lecture 8.1: Modeling with nonlinear systems

Matthew Macauley

Department of Mathematical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

Math 2080, Differential Equations

Epidemiology

The SIR model

Consider an epidemic that spreads through a population, where

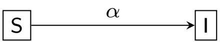
- $S(t) = \#$ susceptible people at time t;
- $I(t) = \#$ infected people at time t;
- $R(t) = \#$ recovered people at time t.

Initially, there are N susceptible (uninfected) people.

Other epidemic models

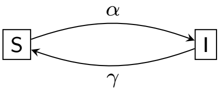
- SI model (e.g., herpes, HIV).

$$\begin{cases} S' = -\alpha SI \\ I' = \alpha SI \end{cases}$$



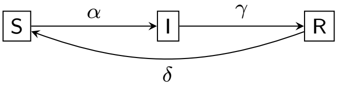
- SIS model. Disease w/o immunity (e.g., chlamydia).

$$\begin{cases} S' = -\alpha SI + \gamma I \\ I' = \alpha SI - \gamma I \end{cases}$$



- SIRS model. Finite-time immunity (e.g., common cold).

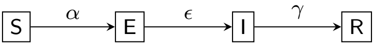
$$\begin{cases} S' = -\alpha SI + \delta R \\ I' = \alpha SI - \gamma I \\ R' = \gamma I - \delta R \end{cases}$$



Other epidemic models

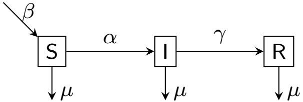
- SEIR model. E = exposed (incubation period, no symptoms).

$$\begin{cases} S' = -\alpha SI \\ E' = \alpha SI - \epsilon E \\ I' = \epsilon E - \gamma I \\ R' = \gamma I \end{cases}$$



- SIR model with birth and death rate.

$$\begin{cases} S' = -\alpha SI + \beta S - \mu S \\ I' = \alpha SI - \gamma I - \mu I \\ R' = \gamma I - \mu R \end{cases}$$



Population dynamics: competing species

Competitive Lotka–Volterra equations

Consider two species competing for a limited food supply.

- $X(t) =$ population of Species 1;
- $Y(t) =$ population of Species 2.

Assume that each species, without the other, would grow logistically.

Population dynamics: predator–prey

Classical Lotka–Volterra equations

Consider two species, one of which depends on the other as a food source:

- $X(t) =$ population of the prey.
- $Y(t) =$ population of the predator.

Assume that in the absense of the other species:

- the prey would grow exponentially;
- the predator would decay exponentially.

Population dynamics: predator–prey

Modified Lotka–Volterra equations

Consider two species, one of which depends on the other as a food source:

- $X(t) =$ population of the prey.
- $Y(t) =$ population of the predator.

Assume that in the absense of the other species:

- the prey would grow logistically;
- the predator would decay exponentially.

Other population models

Immune system vs. infective agent

Let $X(t) =$ population of immune cells, $Y(t) =$ level of infection:

$$\begin{cases} X' = rY - sXY \\ Y' = uY - vXY \end{cases}$$

- $-sXY$: negative effect on immune system from fighting
- $-vXY$: limited effect of immune system in fighting
- rY: immune response is proportionate to infection level

Mutualism

Let $X(t) =$ population of sharks, $Y(t) =$ population of feeder fish:

$$\begin{cases} X' = rX(1 - X/M) + sXY \\ Y' = -uY + vXY \end{cases}$$