Lecture 8.2: Linearization and steady-state analysis

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Math 2080, Differential Equations
Example 1

Consider the following model of two species competing for a limited food supply:

\[
\begin{align*}
X' &= X(1 - X - Y) \\
Y' &= Y(0.75 - Y - 0.5X)
\end{align*}
\]
Nullclines

**Definition**

A nullcline (or isocline with \( c = 0 \)) is a line or curve where \( X' = 0 \) or \( Y' = 0 \).

**Example 1 (cont.)**

Let's find the nullclines of our system:

\[
\begin{align*}
X' &= X(1 - X - Y) \\
Y' &= Y(.75 - Y - .5X)
\end{align*}
\]
Example 1 (cont.)

The 4 steady-states of the following system

\[
\begin{cases}
    X' = X(1 - X - Y) \\
    Y' = Y(0.75 - Y - 0.5X)
\end{cases}
\]

are \((X^*, Y^*) = (0, 0), (1, 0), (0, 0.75), (0.5, 0.5)\). Let’s linearize at \((X^*, Y^*) = (0, 0)\).
Example 1 (cont.)

Let’s linearize this system at a non-zero critical point \((X^*, Y^*)\).

\[
\begin{align*}
X' &= X(1 - X - Y) \\
Y' &= Y(0.75 - Y - 0.5X)
\end{align*}
\]

The first step is to change variables: let \(P = X - X^*\) and \(Q = Y - Y^*\).
Example 1 (cont.)

At a general critical point \((X^*, Y^*)\) we changed variables \((P, Q) = (X - X^*, Y - Y^*)\) to get:

\[
\begin{bmatrix}
P' \\
Q'
\end{bmatrix} = \begin{bmatrix}
1 - 2X^* - Y^* & -X^* \\
-0.5Y^* & 0.75 - 2Y^* - 0.5X^*
\end{bmatrix} \begin{bmatrix}
P \\
Q
\end{bmatrix} + \text{[non-linear terms]}
\]

Recall that our steady-states are \((X^*, Y^*) = (0, 0), (1, 0), (0, 0.75), (0.5, 0.5)\).
Example 2

Consider the following model of two species competing for a limited food supply:

\[
\begin{align*}
X' &= X(1 - X - Y) \\
Y' &= Y(0.8 - 0.6Y - X)
\end{align*}
\]

It is easy to check that there are **four steady-state solutions**: \((0, 0), (1, 0), (0, \frac{4}{3}), (\frac{1}{2}, \frac{1}{2})\).
Four types of dynamics

Competitive Lotka–Volterra equations

Consider the following model of two species competing for a limited food supply:

\[
\begin{align*}
X' &= X(\varepsilon_1 - \sigma_1 X - \alpha_1 Y) \\
Y' &= Y(\varepsilon_2 - \sigma_2 Y - \alpha_2 X)
\end{align*}
\]

- **X-nullclines**: \( X = 0 \) and \( \varepsilon_1 - \sigma_1 X - \alpha_1 Y = 0 \).
- **Y-nullclines**: \( Y = 0 \) and \( \varepsilon_2 - \sigma_2 Y - \alpha_2 X = 0 \).
When might linearization fail?

Consider a $2 \times 2$ matrix with characteristic equation $\lambda^2 - (\text{tr } A)\lambda + \det A = \lambda^2 - p\lambda + q = 0$. 