

Math 4120/6120: Abstract Algebra (Spring 2014)

Midterm 2

NAME:

Instructions: Answer each of the following questions completely. If something is unclear, or if you have any questions, then please ask. Good luck!

1. (10 points) Recall that a permutation $\sigma \in S_n$ is *even* if it can be written as the product of an even number of transpositions. Otherwise, it is *odd*. Let \mathbb{R}^* be the group of nonzero real numbers under multiplication.

Let $\phi: S_n \rightarrow \mathbb{R}^*$ be the homomorphism defined by:
$$\phi(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is an even permutation} \\ -1 & \text{if } \sigma \text{ is an odd permutation} \end{cases}$$

(a) Find $\text{Im}(\phi)$ and $\text{Ker}(\phi)$.

(b) What can we conclude from the Fundamental Homomorphism Theorem? (Be as specific as possible.)

2. (10 points) Answer the following about group actions.

(a) Carefully and completely state the Orbit-Stabilizer theorem.

(b) Suppose a group G of order 35 acts on a set S of size 9. Prove that there must be a fixed point.

3. (20 points) Let H be a subgroup of G .

(a) The quotient group G/H exists if and only if H _____.

(b) The quotient group G/H consists of the set of _____.

(c) Multiplication (i.e., the binary operation) in G/H is defined by _____.

(d) Prove that if G is abelian, then G/H is abelian.

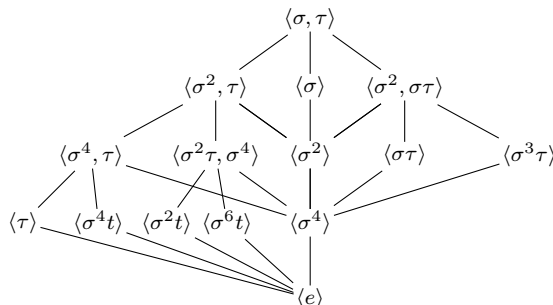
(e) Does the converse to Part (d) hold? If yes, provide a proof. If no, provide a counterexample.

(f) The Correspondence Theorem says that there is a one-to-one correspondence between subgroups of G/H and subgroups of _____.

4. (20 points) The *quasidihedral group* is a nonabelian group of order 16 with the following presentation:

$$QD_8 = \langle \sigma, \tau \mid \sigma^8 = 1, \tau^2 = 1, \sigma\tau = \tau\sigma^3 \rangle.$$

Answer the following questions about $G = QD_8$. You should be able to do all of these from *only* the subgroup lattice of G , shown below. For full credit, completely justify your answers!



- (a) Find the order of all subgroups of G (no justification needed). Do this by writing the order next to each subgroup in the diagram above, or otherwise making it clear from marking up the diagram.
- (b) Given that the abelianization of G is $G/G' \cong V_4$, determine which subgroup of G is the commutator subgroup, G' . [Give generator(s), not just what it is isomorphic to.]
- (c) Given the knowledge that $\langle \sigma^4 \rangle \triangleleft G$, determine what well-known group the quotient $G/\langle \sigma^4 \rangle$ is isomorphic to.
- (d) Other than G , $\langle \sigma^4 \rangle$, and $\langle e \rangle$, there are a few other subgroups that you can be absolutely certain are normal in G . Name these subgroups, and explain how you know they are normal.
- (e) Determine the normalizer $N_G(\langle \sigma^4 \rangle)$.
- (f) What subgroup is $\langle \sigma^4 \tau, \sigma^6 \tau \rangle$?

5. (20 points) Let H be a subgroup of G .

(a) Prove that $H \cong xHx^{-1}$ for any $x \in G$. [*Hint*: Define a map and show it is a bijective homomorphism.]

(b) Prove that if H is the unique subgroup of G of order $|H|$, then H must be normal. You may use the result of Part (a) even if you cannot prove it.

6. (20 points) Answer the following questions about group actions.

(a) Suppose G acts on itself (that is, $S = G$) by conjugation. Then x is a fixed point if and only if _____ . (Be as specific as possible!).

(b) Suppose G acts on itself by right multiplication. Then the stabilizer of any element $x \in G$ is _____ . (Be as specific as possible!).

(c) Let $G = V_4 = \langle v, h \rangle$ act on its set of subgroups, $S = \{ \langle e \rangle, \langle v \rangle, \langle h \rangle, \langle vh \rangle, V_4 \}$, by conjugation. Draw the action diagram. How many orbits are there? Find the stabilizer of each $s \in S$.