1. For each of the following ideals, determine if it is prime and if it is maximal.

   (a) The ideal $I = (x)$ in the polynomial ring $R = \mathbb{Z}[x]$.

   (b) The ideal $I = (x)$ in the polynomial ring $R = \mathbb{R}[x]$.

   (c) The ideal $I = (x, y)$ in the multivariate polynomial ring $R = \mathbb{Z}[x, y]$.

   (d) The ideal $I = (x, y)$ in the multivariate polynomial ring $R = \mathbb{R}[x, y]$.

2. Let R be a commutative ring with 1.

   (a) Prove that R is an integral domain if and only if 0 is a prime ideal.

   (b) Prove that an ideal $P \subseteq R$ is prime if and only if R/P is an integral domain.

   (c) Show that every maximal ideal is prime.

3. Recall that $a, b \in R$ are *associates*, denoted $a \sim b$, if $a \mid b$ and $b \mid a$. Show that $a \sim b$ if and only if $a = bu$ for some unit $u \in R$.

4. Let R be a principal ideal domain (PID). A *common multiple* of $a, b \in R^*$ is an element m such that $a|m$ and $b|m$. Moreover, m is a *least common multiple* (LCM) if $m|n$ for any other common multiple n of a and b.

   (a) Prove that any $a, b \in R^*$ have an LCM.

   (b) Prove that an LCM of a and b is unique up to multiplication of associates, and can be characterized as a generator of the (principal) ideal $I := (a) \cap (b)$.

5. For any $x = r + s\sqrt{m} \in \mathbb{Q}(\sqrt{m})$, define the *norm* of x to be $N(x) = r^2 - ms^2$.

   (a) Show that $N(xy) = N(x)N(y)$.

   (b) Show that $N(x) \in \mathbb{Z}$ if $x \in R_m$.

   (c) Show that $u \in U(R_m)$ if and only if $|N(u)| = 1$.

   (d) Show that $U(R_{-1}) = \{\pm 1, \pm i\}$, $U(R_{-3}) = \{\pm 1, \pm(1 \pm \sqrt{3})/2\}$, and $U(R_m) = \{\pm 1\}$ for all other negative square-free $m \in \mathbb{Z}$.

6. Let $R = \mathbb{Z}_{10}$ and $D = \{0, 2, 4, 6, 8\} \subset R$. Find the field of fractions of D in R.