Lecture 1.4: Group presentations

Matthew Macauley

Department of Mathematical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

Math 4120, Modern Algebra

Overview

Recall that our informal definition of a group was a collection of actions that obeyed Rules 1–4. This is not the ordinary definition of a group.

Over the course of the next few lectures, we will be working toward introducing the standard (and more formal) definition of a group.

Along the way, we will learn about some helpful tools to get us there.

In this lecture, we will introduce group presentations, an algebraic device to concisely describe groups by their generators and relations.

For example, the following is a presentation for a group that we are familiar with:

$$G = \langle a, b \mid a^2 = 1,\ b^2 = 1,\ ab = ba \rangle .$$

Do you recognize this group?

More on Cayley diagrams

Recall that arrows in a Cayley diagram represent one choice of generators of the group. In particular, all arrows of a fixed color correspond to the same generator.

Our choice of generators influenced the resulting Cayley diagram!

When we have been drawing Cayley diagrams, we have been doing one of two things with the nodes:

1. Labeling the nodes with configurations of a thing we are acting on.
2. Leaving the nodes unlabeled (this is the "abstract Cayley diagram").

There is a 3rd thing we can do with the nodes, motivated by the fact that every path in the Cayley diagram represents an action of the group:

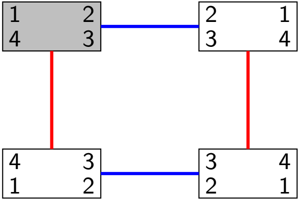
3. Label the nodes with actions (this is called a "diagram of actions").

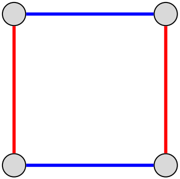
Motivating idea

If we distinguish one node as the "unscrambled" configuration and label that with the identity action, then we can label each remaining node with the action that it takes to reach it from the unscrambled state.

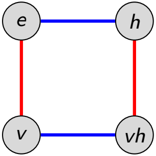
An example: The Klein 4-group

Recall the "rectangle puzzle." If we use horizontal flip (h) and vertical flip (v) as generators, then here is the Cayley diagram labeled by configurations (left), and unlabeled Cayley diagram (right):





Let's apply the steps to the abstract Cayley diagram for V_4, using the upper-left node as the "unscrambled configuration":



Note that we could also have labeled the node in the lower right corner as hv, as well.

How to label nodes with actions

Let's summarize the process that we just did.

Node labeling algorithm

The following steps transform a Cayley diagram into one that focuses on the group's actions.

(i) Choose a node as our initial reference point; label it e. (This will correspond to our "identity action.")

(ii) Relabel each remaining node in the diagram with a path that leads there from node e. (If there is more than one path, pick any one; shorter is better.)

(iii) Distinguish arrows of the same type in some way (color them, label them, dashed vs. solid, etc.)

Our convention will be to label the nodes with sequences of generators, so that reading the sequence from left to right indicates the appropriate path.

Warning!

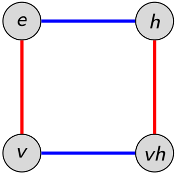
Some authors use the opposite convention, motivated by "function composition."

A "group calculator"

One neat thing about Cayley diagrams with nodes labeled by actions is that they act as a "group calculator".

For example, if we want to know what a particular sequence is equal to, we can just chase the sequence through the Cayley graph, starting at e.

Let's try one. In V_4, what is the action $hhhvhvvhv$ equal to?



We see that $hhhvhvvhv = h$. A more condensed way to write this is

$$hhhvhvvhv = h^3vhv^2hv = h.$$

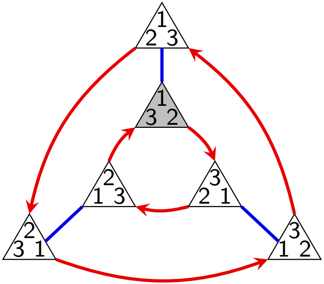
A concise way to describe V_4 is by the following group presentation (more on this later):

$$V_4 = \langle v, h \mid v^2 = e, h^2 = e, vh = hv \rangle .$$

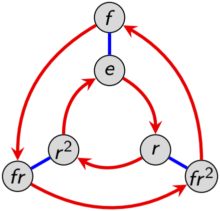
Another familiar example: D_3

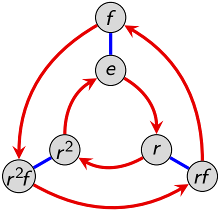
Recall the "triangle puzzle" group $G = \langle r, f \rangle$, generated by a clockwise 120° rotation r, and a horizontal flip f.

Let's take the shaded triangle to be the "unscrambled configuration."



Here are two different ways (of many!) that we can label the nodes with actions:





The following is one (of many!) presentations for this group:

$$D_3 = \langle r, f \mid r^3 = e,\ f^2 = e,\ r^2 f = fr \rangle .$$

Group presentations

Initially, we wrote $G = \langle h, v \rangle$ to say that "G is generated by the elements h and v."

All this tells us is that h and v generate G, but not how they generate G.

If we want to be more precise, we use a group presentation of the following form:

$$G = \left\langle \text{generators} \;\middle|\; \text{relations} \right\rangle$$

The vertical bar can be thought of as meaning "subject to".

For example, the following is a presentation for V_4:

$$V_4 = \langle a, b \mid a^2 = e,\ b^2 = e,\ ab = ba \rangle \,.$$

Caveat!

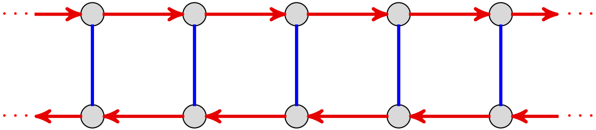
Just because there are elements in a group that "satisfy" the relations above does *not* mean that it is V_4.

For example, the trivial group $G = \{e\}$ satisfies the above presentation; just take $a = e$ and $b = e$.

Loosely speaking, the above presentation tells us that V_4 is the "largest group" that satisfies these relations. (More on this when we study quotients.)

Group presentations

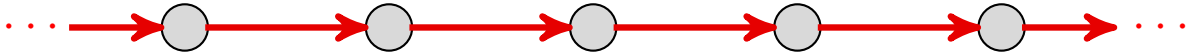
Recall the frieze group from Chapter 3 that had the following Cayley diagram:



One presentation of this group is

$$G = \langle t, f \mid f^2 = e, tft = f \rangle .$$

Here is the Cayley diagram of another frieze group:



It has presentation

$$G = \langle a \mid \quad \rangle .$$

That is, "one generator subject to *no relations*."

Group presentations

Due to the aforementioned caveat, and a few other technicalities, the study of group presentations is a topic usually relegated to graduate-level algebra classes.

However, they are often introduced in an undergraduate algebra class because *they are very useful*, even if the intricate details are harmlessly swept under the rug.

The problem (called the word problem) of determining what a mystery group is from a presentation is actually computationally unsolvable! In fact, it is equivalent to the famous "halting problem" in computer science!

For (mostly) amusement, what group do you think the following presentation describes?

$$G = \langle a, b \mid ab = b^2 a,\ ba = a^2 b \rangle.$$

Surprisingly, this is the trivial group $G = \{e\}$!