Lecture 6.4: Galois groups

Matthew Macauley

Department of Mathematical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

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The Galois group of a polynomial

Definition

Let $f \in \mathbb{Z}[x]$ be a polynomial, with roots $r_1, \ldots, r_n$. The splitting field of f is the field

$$\mathbb{Q}(r_1, \ldots, r_n)\,.$$

The splitting field F of $f(x)$ has several equivalent characterizations:

- the smallest field that contains all of the roots of $f(x)$;
- the smallest field in which $f(x)$ splits into linear factors:

$$f(x) = (x - r_1)(x - r_2) \cdots (x - r_n) \in F[x]\,.$$

Recall that the Galois group of an extension $F \supseteq \mathbb{Q}$ is the group of automorphisms of F, denoted $\mathrm{Gal}(F)$.

Definition

The Galois group of a polynomial $f(x)$ is the Galois group of its splitting field, denoted $\mathrm{Gal}(f(x))$.

A few examples of Galois groups

- The polynomial $x^2 - 2$ splits in $\mathbb{Q}(\sqrt{2})$, so
$$\mathrm{Gal}(x^2 - 2) = \mathrm{Gal}(\mathbb{Q}(\sqrt{2})) \cong C_2\,.$$
- The polynomial $x^2 + 1$ splits in $\mathbb{Q}(i)$, so
$$\mathrm{Gal}(x^2 + 1) = \mathrm{Gal}(\mathbb{Q}(i)) \cong C_2\,.$$
- The polynomial $x^2 + x + 1$ splits in $\mathbb{Q}(\zeta)$, where $\zeta = e^{2\pi i/3}$, so
$$\mathrm{Gal}(x^2 + x + 1) = \mathrm{Gal}(\mathbb{Q}(\zeta)) \cong C_2\,.$$
- The polynomial $x^3 - 1 = (x-1)(x^2 + x + 1)$ also splits in $\mathbb{Q}(\zeta)$, so
$$\mathrm{Gal}(x^3 - 1) = \mathrm{Gal}(\mathbb{Q}(\zeta)) \cong C_2\,.$$
- The polynomial $x^4 - x^2 - 2 = (x^2 - 2)(x^2 + 1)$ splits in $\mathbb{Q}(\sqrt{2}, i)$, so
$$\mathrm{Gal}(x^4 - x^2 - 2) = \mathrm{Gal}(\mathbb{Q}(\sqrt{2}, i)) \cong V_4\,.$$
- The polynomial $x^4 - 5x^2 + 6 = (x^2 - 2)(x^2 - 3)$ splits in $\mathbb{Q}(\sqrt{2}, \sqrt{3})$, so
$$\mathrm{Gal}(x^4 - 5x^2 + 6) = \mathrm{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{3})) \cong V_4\,.$$
- The polynomial $x^3 - 2$ splits in $\mathbb{Q}(\zeta, \sqrt[3]{2})$, so
$$\mathrm{Gal}(x^3 - 2) = \mathrm{Gal}(\mathbb{Q}(\zeta, \sqrt[3]{2})) \cong D_3 \text{ ???}$$

The tower law of field extensions

Recall that if we had a chain of subgroups $K \leq H \leq G$, then the index satisfies a tower law: $[G : K] = [G : H][H : K]$.

Not surprisingly, the degree of field extensions obeys a similar tower law:

Theorem (Tower law)

For any chain of field extensions, $F \subset E \subset K$,

$$[K : F] = [K : E][E : F]\,.$$

We have already observed this in our subfield lattices:

$$[\mathbb{Q}(\sqrt{2},\sqrt{3}) : \mathbb{Q}] = [\underbrace{\mathbb{Q}(\sqrt{2},\sqrt{3}) : \mathbb{Q}(\sqrt{2})}_{\text{min. poly: } x^2-3}][\underbrace{\mathbb{Q}(\sqrt{2}) : \mathbb{Q}}_{\text{min. poly: } x^2-2}] = 2 \cdot 2 = 4\,.$$

Here is another example:

$$[\mathbb{Q}(\zeta,\sqrt[3]{2}) : \mathbb{Q}] = [\underbrace{\mathbb{Q}(\zeta,\sqrt[3]{2}) : \mathbb{Q}(\sqrt[3]{2})}_{\text{min. poly: } x^2+x+1}][\underbrace{\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}}_{\text{min. poly: } x^3-2}] = 2 \cdot 3 = 6\,.$$

Primitive elements

Primitive element theorem

If F is an extension of $\mathbb{Q}$ with $[F:\mathbb{Q}]<\infty$, then F has a primitive element: some $\alpha\notin\mathbb{Q}$ for which $F=\mathbb{Q}(\alpha)$.

How do we find a primitive element α of $F=\mathbb{Q}(\zeta,\sqrt[3]{2})=\mathbb{Q}(i\sqrt{3},\sqrt[3]{2})$?

Let's try $\alpha=i\sqrt{3}\sqrt[3]{2}\in F$. Clearly, $[\mathbb{Q}(\alpha):\mathbb{Q}]\leq 6$. Observe that

$$\alpha^2=-3\sqrt[3]{4},\quad \alpha^3=-6i\sqrt{3},\quad \alpha^4=-18\sqrt[3]{2},\quad \alpha^5=18i\sqrt[3]{4}\sqrt{3},\quad \alpha^6=-108.$$

Thus, α is a root of x^6+108. The following are equivalent (why?):

(i) α is a primitive element of F;

(ii) $[\mathbb{Q}(\alpha):\mathbb{Q}]=6$;

(iii) the minimal polynomial $m(x)$ of α has degree 6;

(iv) x^6+108 is irreducible (and hence must be $m(x)$).

In fact, $[\mathbb{Q}(\alpha):\mathbb{Q}]=6$ holds because both 2 and 3 divide $[\mathbb{Q}(\alpha):\mathbb{Q}]$:

$$[\mathbb{Q}(\alpha):\mathbb{Q}]=[\mathbb{Q}(\alpha):\mathbb{Q}(i\sqrt{3})]\underbrace{[\mathbb{Q}(i\sqrt{3}):\mathbb{Q}]}_{=2},\qquad [\mathbb{Q}(\alpha):\mathbb{Q}]=[\mathbb{Q}(\alpha):\mathbb{Q}(\sqrt[3]{2})]\underbrace{[\mathbb{Q}(\sqrt[3]{2}):\mathbb{Q}]}_{=3}.$$

An example: The Galois group of $x^4 - 5x^2 + 6$

The polynomial $f(x) = (x^2 - 2)(x^2 - 3) = x^4 - 5x^2 + 6$ has splitting field $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.

We already know that its Galois group should be V_4. Let's compute it explicitly; this will help us understand it better.

We need to determine all automorphisms ϕ of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$. We know:

- ϕ is determined by where it sends the basis elements $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$.
- ϕ must fix 1.
- If we know where ϕ sends two of $\{\sqrt{2}, \sqrt{3}, \sqrt{6}\}$, then we know where it sends the third, because
$$\phi(\sqrt{6}) = \phi(\sqrt{2}\sqrt{3}) = \phi(\sqrt{2})\,\phi(\sqrt{3})\,.$$

In addition to the identity automorphism e, we have

$$\begin{cases} \phi_2(\sqrt{2}) = -\sqrt{2} \\ \phi_2(\sqrt{3}) = \quad \sqrt{3} \end{cases} \qquad \begin{cases} \phi_3(\sqrt{2}) = \quad \sqrt{2} \\ \phi_3(\sqrt{3}) = -\sqrt{3} \end{cases} \qquad \begin{cases} \phi_4(\sqrt{2}) = -\sqrt{2} \\ \phi_4(\sqrt{3}) = -\sqrt{3} \end{cases}$$

Question

What goes wrong if we try to make $\phi(\sqrt{2}) = \sqrt{3}$?

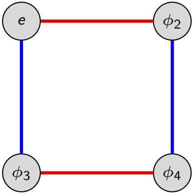
An example: The Galois group of $x^4 - 5x^2 + 6$

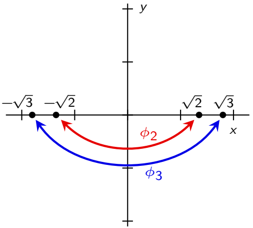
There are 4 automorphisms of $F = \mathbb{Q}(\sqrt{2}, \sqrt{3})$, the splitting field of $x^4 - 5x^2 + 6$:

$$\begin{aligned}
e\colon a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} &\longmapsto a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} \\
\phi_2\colon a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} &\longmapsto a - b\sqrt{2} + c\sqrt{3} - d\sqrt{6} \\
\phi_3\colon a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} &\longmapsto a + b\sqrt{2} - c\sqrt{3} - d\sqrt{6} \\
\phi_4\colon a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} &\longmapsto a - b\sqrt{2} - c\sqrt{3} + d\sqrt{6}
\end{aligned}$$

They form the Galois group of $x^4 - 5x^2 + 6$. The multiplication table and Cayley diagram are shown below.

	e	ϕ_2	ϕ_3	ϕ_4
e	e	ϕ_2	ϕ_3	ϕ_4
ϕ_2	ϕ_2	e	ϕ_4	ϕ_3
ϕ_3	ϕ_3	ϕ_4	e	ϕ_2
ϕ_4	ϕ_4	ϕ_3	ϕ_2	e





Exercise

Show that $\alpha = \sqrt{2} + \sqrt{3}$ is a primitive element of F, i.e., $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.