Topics: Vector spaces, linear independence, and bases

1. For each of the following sets, determine if it is a *vector space* over $\mathbb{R}$. If it is, give an explicit *basis* and compute its *dimension*. If it isn't, explain why not by giving an example of how one of the vector space properties fails.

   (a) The set of points in $\mathbb{R}^3$ with $x = 0$.

   (b) The set of points in $\mathbb{R}^2$ with $x = y$.

   (c) The set of points in $\mathbb{R}^3$ with $x = y$.

   (d) The set of points in $\mathbb{R}^3$ with $z \geq 0$.

   (e) The plane in $\mathbb{R}^3$ defined by the equation $z = 2x - 3y$.

   (f) The set of unit vectors in $\mathbb{R}^2$.

   (g) The set of polynomials of degree n.

   (h) The set $\mathbb{R}_n[x]$ of polynomials of degree at most n.

   (i) The set of polynomials of degree at most n, with only even-powers of x.

   (j) The set $\mathrm{Per}_{2\pi}(\mathbb{R})$ of piecewise continuous functions, i.e., f such that $f(x) = f(x+2\pi)$.

   (k) $\mathbb{C} := \{a + bi \mid a, b \in \mathbb{R}\}$.

2. Let v_1, v_2, w be three *linearly independent* vectors in $\mathbb{R}^3$. That is, they do not all lie on the same plane. For each of the following (infinite) set of vectors, carefully sketch it in $\mathbb{R}^3$, and determine whether or not it is a vector space (i.e., a *subspace* of $\mathbb{R}^3$). Explain your reasoning.

   (a) $\{Cv_1 \mid C \in \mathbb{R}\}$

   (b) $\{Cv_1 + w \mid C \in \mathbb{R}\}$

   (c) $\{C_1v_1 + C_2v_2 \mid C_1, C_2 \in \mathbb{R}\}$

   (d) $\{C_1v_1 + C_2v_2 + w \mid C_1, C_2 \in \mathbb{R}\}$

3. Find the general solution to each of the following ODEs. Then, decide whether or not the set of solutions form a vector space. Explain your reasoning. Compare your answers to the previous problem. Recall that the general solution has the form $y(t) = y_h(t) + y_p(t)$.

   (a) $y' - 2y = 0$

   (b) $y' - 2y = 1$

   (c) $y'' + 4y = 0$

   (d) $y'' + 4y = e^{3t}$

4. For each of the following pairs of vectors $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$ in (a)–(e), carry out the following steps:

   (i) The lines through v_1 and v_2 generate a grid (of parallelograms) on the xy-plane. Sketch v_1, v_2, and this grid.

   (ii) Find the area of one of the parallelgrams by computing the determinant of the matrix $\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$. Is this matrix invertible?

   (iii) Determine whether $\{v_1, v_2\}$ is a basis of $\mathbb{R}^2$.

(a) $v_1 = (1, 0)$, $v_2 = (0, 1)$

(b) $v_1 = (2, 0)$, $v_2 = (0, 2)$

(c) $v_1 = (\frac{1}{2}, \frac{1}{2})$, $v_2 = (\frac{1}{2}, -\frac{1}{2})$

(d) $v_1 = (1, 1)$, $v_2 = (1, 2)$

(e) $v_1 = (1, 2)$, $v_2 = (1, 1)$

(f) $v_1 = (2, -1)$, $v_2 = (-4, 2)$

Summarize your conclusions in a sentence or two.

5. For each of the following triples of vectors $v_1 = (x_1, y_1, z_1)$, $v_2 = (x_2, y_2, z_2)$, and $v_3 = (x_3, y_3, z_3)$, carry out the following steps:

   (i) Sketch v_1, v_2, and v_3 in $\mathbb{R}^3$.

   (ii) Use a computer to calculate the determinant of $\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$. Is it invertible?

   (iii) The lines through v_1, v_2, and v_3 generate a tessellation (of parallelepipeds) in $\mathbb{R}^3$. What do you think the volume of each parallelepiped is?

   (iv) Describe in words (e.g., line, plane, all of $\mathbb{R}^3$) the subspace $\mathrm{Span}\{v_1, v_2, v_3\}$. Is $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$?

   (a) $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$, $v_3 = (0, 0, 1)$

   (b) $v_1 = (2, 0, 0)$, $v_2 = (0, 2, 0)$, $v_3 = (0, 0, 2)$

   (c) $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 1)$, $v_3 = (3, 1, 1)$

   (d) $v_1 = (1, 0, 0)$, $v_2 = (0, 2, -1)$, $v_3 = (1, 1, 1)$

   Summarize your conclusions in a sentence or two.

6. For each of the following, a vector space V is given, along with a finite set $S \subset V$. Denote the subspace of V *spanned* by S as $\mathrm{Span}(S)$. Find an explicit basis for $\mathrm{Span}(S)$ and compute its dimension.

   (a) $V = \mathbb{R}^3$, $S = \{(1, 0, 0), (0, 1, 1), (1, 1, 1), (3, 1, 1)\}$.

   (b) $V = \mathbb{R}^2$, $S = \{(1, 0), (0, 1), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2})\}$.

   (c) $V = \mathcal{C}^\infty(\mathbb{R})$, $S = \{e^{3x}, e^{-3x}, \cosh 3x, \sinh 3x\}$.