TOPICS: CAUCHY-EULER EQUATIONS AND POWER SERIES SOLUTIONS TO ODES

1. For each of the *Cauchy-Euler equations* below, look for a solution of the form $y(x) = x^r$, and plug this back in and find r. Find a basis of the solution space consisting of two real-valued functions, and use this to write the general solution.

   (a) $x^2y'' - xy' - 3y = 0$

   (b) $x^2y'' - xy' + 5y = 0$

   (c) $x^2y'' - 3xy' + 4y = 0$

2. Write each of the following as a single series of the form $\sum f(n)x^n$. That is, $f(n)$ is the coefficient of x^n. You may need to additionally "pull out" the first term(s) from one of the sums.

   (a) $\displaystyle\sum_{n=0}^{5} x^{n-1}$

   (b) $\displaystyle\sum_{n=0}^{5} x^{n+1}$

   (c) $\displaystyle\sum_{n=0}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n$

   (d) $\displaystyle\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^{n+1}$

   (e) $\displaystyle 5\sum_{n=0}^{\infty} n(n-1)x^{n-2} + \sum_{n=0}^{\infty} nx^{n-1} - \sum_{n=0}^{\infty} x^n.$

3. Consider the ODE $y'' - 2xy' + 10y = 0$. Note that unlike the equation in the first problem, there will not longer be a simple solution of the form x^r. However, we know that the solution space is 2-dimensional, and most "nice" functions can be written as a power series. Therefore, we'll look for a solution of the form $y(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$.

   (a) Plug $y(x)$ back into the ODE and find a recurrence relation for a_{n+2} in terms of a_n and a_{n+1}.

   (b) Explicitly write out the coefficients a_n for $n \leq 9$, in terms of a_0 and a_1. Write down formulas for a_{2n} and a_{2n+1} in terms of a_0 and a_1.

   (c) Since the solution space to this ODE is 2-dimensional, the general solution you found in Part (a) can be written as $y(x) = C_0y_0(x) + C_1y_1(x)$. Find such a *basis*, $\{y_0(x), y_1(x)\}$.

   (d) Find a non-zero *polynomial* solution. [*Hint*: Make a good choice for a_0 and a_1.]

   (e) Are there any other polynomial solutions, excluding scalar multiples of the one you found in (d)? Why or why not?

   (f) Consider the initial value problem

   $$y'' - 2xy' + 10y = 0\,, \qquad y(0) = x_0\,, \quad y'(0) = v_0\,.$$

   What are x_0 and v_0 in terms of the coefficients a_n?

4. The differential equation $(1-x^2)y'' - 2xy' + \nu(\nu+1)y = 0$, where ν is a constant, is known as *Legendre's equation.* It is used for modeling sperically symmetric potentials in the theory of Newtonian gravitation and in electricity and magnetism.

   (a) Assume that the general solution has the form $y(t) = \sum_{n=0}^{\infty} a_n x^n$, and find the recursion formula for a_{n+2} in terms of a_n and a_{n+1}.

   (b) Use the recursion formula to determine a_n in terms of a_0 and a_1, for $2 \leq n \leq 9$.

   (c) For each $\nu \in \mathbb{N}$, there will be a single (up to scalar multiples) nonzero polynomial solution $P_\nu(x)$, called the *Legendre polynomial* of degree ν. Find the Legendre polynomial of degree $\nu = 3$.

   (d) Find a basis for the solution space to $(1-x^2)y'' - 2xy' + 12y = 0$.

5. The differential equation $y'' - xy = 0$ is called *Airy's equation*, and is used in physics to model the refraction of light.

   (a) Assume a power series solution, and find the recurrence relation of the coefficients. [*Hint*: When shifting the indices, one way is to let $m = n-3$, then factor out x^{n+1} and find a_{n+3} in terms of a_n. Alternatively, you can find a_{n+2} in terms of a_{n-1}.]

   (b) Show that $a_2 = 0$. [*Hint*: the two series for y'' and xy don't "start" at the same power of x, but for any solution, each term must be zero. (Why?)]

   (c) Find the particular solution when $y(0) = 1$, $y'(0) = 0$, as well as the particular solution when $y(0) = 0$, $y'(0) = 1$.

6. Consider the following initial value problem:
$$y''' - y = 0\,, \qquad y(0) = 1\,, \qquad y'(0) = 0\,, \qquad y''(0) = 0\,.$$

   (a) Assume there is a solution of the form $y(t) = e^{rt}$, and plug this back in and solve for r. Use this to write the general solution. [*Hint*: The equation $r^3 = 1$ has 3 distinct solutions over $\mathbb{C}$, called the *3rd roots of unity*: $r_1 = e^{0\pi i/3} = 1$, $r_2 = e^{2\pi i/3}$, and $r_2 = e^{4\pi i/3}$.]

   (b) From here, you could solve the initial value problem by plugging in the initial conditions, but you'll get a system of three equations with three unknowns, and involving complex numbers. Here's another way: look for a power series solution, $y(t) = \sum_{n=0}^{\infty} a_n t^n$. Plug this back into the ODE and find the recurrence relation for the coefficients.

   (c) Compute a_n for $n \leq 10$, and once you see the pattern, write down the general solution to the ODE as a power series. [*Hint*: It should look familiar!]

   (d) Write down a *basis* for the solution space.

   (e) Plug in the initial conditions and find the particular solution to the IVP.

   (f) Solve a similar initial value problem:
$$y''' - y = 0\,, \qquad y(0) = 1\,, \qquad y'(0) = 1\,, \qquad y''(0) = 1\,.$$