Topics: The Frobenius method and Bessel's equation

1. The differential equation $(1-x^2)y'' - xy' + p^2y = 0$, where p is a constant, is known as *Chebyshev's equation*. It can be rewritten in the form
$$y'' + P(x)y' + Q(x)y = 0\,, \qquad P(x) = -\frac{x}{1-x^2}\,, \quad Q(x) = \frac{p^2}{1-x^2}\,.$$
   (a) If $P(x)$ and $Q(x)$ are represented as a power series about $x_0 = 0$, what is the radius of convergence of these power series?

   (b) Assume that the general solution has the form $\sum_{n=0}^{\infty} a_n x^n$, and find a recurrence for a_{n+2} in terms of a_n. [*Hint*: Before plugging back in, multiply through by $1-x^2$.]

   (c) Use the recurrence to determine a_n in terms of a_0 and a_1, for $2 \leq n \leq 9$.

   (d) For each $p \in \mathbb{N}$, there is a unique polynomial solution $T_p(x)$ known as the *Chebyshev polynomial* of degree p. Find $T_3(x)$.

2. For each of the following ODEs, determine whether $x = 0$ is an ordinary or singular point. If it is singular, determine whether it is regular or not. (Remember, first write each ODE in the form $y'' + P(x)y' + Q(x)y = 0$.)

   (a) $y'' + xy' + (1-x^2)y = 0$

   (b) $y'' + (1/x)y' + (1-(1/x^2))y = 0$.

   (c) $x^2y'' + 2xy' + (\cos x)y = 0$.

   (d) $x^3y'' + 2xy' + (\cos x)y = 0$.

3. Consider the differential equation $3xy'' + y' + y = 0$. Since $x_0 = 0$ is a regular singular point, there is a generalized power series solution of the form $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$.

   (a) Determine the indicial equation (solve for r) and the recurrence relation for the coefficients.

   (b) Find two linearly independent generalized power series solutions (i.e., a *basis* for the solution space).

   (c) Determine the radius of convergence of each of these solutions. [*Hint*: First compute the radius of convergence of $xP(x)$ and $x^2Q(x)$ and apply Frobenius].

4. Consider the differential equation $2xy'' + y' + xy = 0$. Since $x_0 = 0$ is a regular singular point, there is a solution of the form $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$.

   (a) Determine the indicial equation (solve for r) and the recursion formula.

   (b) Find a basis for the solution space and use this to write the general solution.

   (c) What is the radius of convergence of each of these two linearly independent solutions?

5. Consider the differential equation $xy'' + 2y' - xy = 0$.
   (a) Show that $x = 0$ is a regular singular point.
   (b) Show that if $a_0 = 0$, then $r = -1$ is one solution for the indicial equation.
   (c) For $r = -1$ and $a_0 = 0$, find the recurrence relation for a_{n+2} in terms of a_n.
   (d) Still assuming that $a_0 = 0$, write the solution from (b) as a generalized power series.
   (e) Write this solution using a standard hyperbolic trigonometric function.

6. Consider the ODE $y'' + e^{-x}y = 0$. Change variables by setting $t = 2e^{-x/2}$, which will reduce this ODE to a Bessel's equation. Using the known solution to Bessel's equation, back-substitute to determine the solution $y(x)$ to the original ODE.