Topics: Diffusion and the heat equation

In the process of solving these problems, you will encounter several ODEs, Sturm-Liouville problems, and Fourier series, many of which you have encountered before (especially on previous homeworks). You do not need to re-derive the solutions of anything you have previously solved.

1. We will solve the following B/IVP for $u(x,t)$, defined for $0 \leq x \leq 1$ and $t \geq 0$:

$$u_t = c^2 u_{xx}, \qquad u(0,t) = u(1,t) = 0, \qquad u(x,0) = 5\sin \pi x + 3 \sin 2\pi x.$$

This PDE is called the *heat equation*. It is homogeneous.

(a) Briefly describe, and sketch, a physical situation which this models. Be sure to explain the effect of both boundary conditions and the initial condition.

(b) Assume that $u(x,t) = f(x)g(t)$. Find u_t and u_{xx}. Also, determine the boundary conditions for $f(x)$ from the boundary conditions for $u(x,t)$.

(c) Plug $u = fg$ back into the PDE and divide both sides by $c^2 fg$, i.e., "separate variables." Briefly justify why this quantity must be a constant. Call this constant $-\lambda$. Write down two ODEs: One for $g(t)$, and a Sturm-Lioville problem for $f(x)$.

(d) Determine $f(x)$, λ, and $g(t)$.

(e) Using your solution to Part (d) and linearity (in physical terms, the principle of superposition), find the general solution to this PDE.

(f) Solve the resulting initial value problem. That is, find the particular solution $u(x,t)$ that satisfies the boundary and initial conditions.

(g) What is the steady-state solution, $\lim_{t\to\infty} u(x,t)$? Explain the physical interpretation.

2. Solve the following PDE:

$$u_t = c^2 u_{xx}\,, \qquad u_x(0,t) = u(\pi,t) = 0, \qquad u(x,0) = 40 + 6\cos\tfrac{5x}{2}\,.$$

Make sure you describe and sketch the physical situation which this models, and determine the steady-state solution. Also, make it clear what Sturm-Liouville problem arises.

3. Consider the following PDE with inhomogeneous boundary conditions:

$$u_t = c^2 u_{xx}\,, \qquad u_x(0,t) = 0\,, \quad u(1,t) = 32\,, \qquad u(x,0) = 72 + 6\cos\tfrac{5x}{2}\,.$$

(a) Make the substitution $v(x,t) = u(x,t) - 32$, and re-write the PDE above, including the boundary and initial conditions, in terms of v instead of u.

(b) Briefly describe, and sketch, a physical situation which this models. Be sure to explain the effect of both boundary conditions and the initial condition.

(c) Solve the PDE for $u(x,t)$. You should use the results of the previous problem.

(d) What is the steady-state solution? Give a physical interpretation of this.

4. Consider the following PDE, subject to *periodic boundary conditions*:

$$u_t = c^2 u_{\theta\theta}, \qquad u(\theta + 2\pi, t) = u(\theta, t), \qquad u(\theta, 0) = 2 + 4\sin 3\theta - \cos 5\theta.$$

Briefly describe, and sketch, a physical situation which this models. Solve this PDE via steps (a)–(g) from the first problem. You do not need to repeat any steps that are identical so something you've done before, as long as you point that out. However, you have *not* seen this type of Sturm-Liouville problem before.

5. Let $u(x, t)$ be the temperature of a bar of length 10, at position x and time t (in hours). Suppose that the left endpoint of the bar is not insulated, but the right endpoint is fully insulated, and the bar is sitting in a 70° room. Moreover, suppose that initially, the temperature increases linearly from 70° at the left endpoint to 80° at the other end. Finally, suppose the interior of the bar is poorly insulated, so heat can escape.

   (a) Suppose that heat escapes at a constant rate of 1° per hour. Write (but do not solve) an initial/boundary value problem for $u(x, t)$ that could model this situation.

   (b) A more realistic situation would be for heat to escape not at a constant rate, but at a rate proportional to the *difference* between the temperature of the bar and the ambient temperature of the room. Write an initial/boundary value problem for $u(x, t)$ that could model this situation. What is the steady-state solution and why?

6. Consider the following B/IVP for the heat equation:

$$u_t = c^2 u_{xx}, \qquad u(0, t) = 0, \quad u_x(1, t) = \gamma\, u(1, t), \qquad u(x, 0) = h(x)\,,$$

where γ is constant and $h(x)$ is an arbitrary function.

   (a) Describe a physical situation that this models. Be sure to describe the impact of the initial condition, *both* boundary conditions, and the constant γ. [*Hint*: First consider the special case when $\gamma = 0$. How does this compare to when $\gamma \approx 0$?]

   (b) What is the steady-state solution, and why? (Use your physical intuition.)

   (c) Assume that $u(x, t) = f(x)g(t)$, and plug this back in to get a familiar ODE for $g(t)$, and a Sturm-Liouville problem for $f(x)$.

   (d) Solve the Sturm-Liouville problem. You won't get a closed form for the eigenvalues, but you can write them as $\lambda_n = \omega_n^2$, where $\{\omega_n \mid n = 1, 2, \dots\}$ are the non-negative solutions to a simple algebraic equation. Write down this equation and graph it, clearly marking the first few solutions on the graph.

   (e) Find the general solution to the original boundary value problem.

   (f) The particular solution satisfying the initial and boundary conditions can be found by plugging in $t = 0$ and using the initial condition to determine the b_n. You do not need to do this, but explain the process in a few steps in enough detail that an educated reader (e.g., one of your classmates) could figure out how to do this from your explanation. Your answer should involve the orthogonality of the eigenfunctions and the associated inner product.