TOPICS: THE TRANSPORT, WAVE, AND SCHRÖDINGER EQUATION

In the process of solving these problems, you will encounter several ODEs, Sturm-Liouville problems, PDEs, and Fourier series, many of which you have encountered before. You do not need to re-derive the solutions of anything you have previously solved.

1. Recall from ODEs that $y' = -\lambda y$ models exponential decay. The equation

$$u_t + cu_x = -\lambda u$$

is often called the *convection-decay* equation, and it could model the concentration of a radioactive chemical dissolved in water that flows at speed c. Find the general solution to this PDE by making the following change of variables

$$\xi = x - ct, \qquad \tau = t.$$

2. Consider the following B/IVP for the wave equation, defined for $0 \leq x \leq 1$ and $t \geq 0$:

$$u_{tt} = c^2 u_{xx}, \qquad u(0,t) = u(1,t) = 0, \qquad u(x,0) = x(1-x), \quad u_t(x,0) = 0.$$

   (a) Briefly describe, and sketch, a physical situation which this models. Be sure to explain the effect of both boundary conditions and both initial conditions.

   (b) Assume that $u(x,t) = f(x)g(t)$. Find u_t and u_{xx}. Also, determine the boundary conditions for $f(x)$ from the boundary conditions for $u(x,t)$, and one initial condition for $g(t)$. (The three BC/ICs equal to zero will give you these.)

   (c) Plug $u = fg$ back into the PDE and divide both sides by $c^2 fg$, i.e., "separate variables." Briefly justify why this quantity must be a constant. Call this constant $-\lambda$. Write down two ODEs: One for $g(t)$ with one initial condition, and a Sturm-Lioville problem for $f(x)$.

   (d) Determine $f(x)$, λ, and $g(t)$.

   (e) Using your solution to Part (d) and the principle of superposition, find the general solution to this PDE.

   (f) Use the last initial condition, $u(x,0) = x(1-x)$, to find the particular solution that satisfies the initial and boundary conditions.

   (g) What is the long-term behavior of this system? Give both a mathematical and physical justification.

3. We will solve for the function $u(x,t)$, defined for $0 \leq x \leq 1$ and $t \geq 0$, which satisfies the following conditions:

$$u_{tt} = c^2 u_{xx}, \qquad u(0,t) = u(1,t) = 0, \qquad u(x,0) = 0, \quad u_t(x,0) = x(1-x).$$

The only difference between this and the previous equation are the initial conditions. Carry out each step (a)–(g) from the previous problem, unless it is idential (many of them will be), in which case you should say "same as the previous problem" instead.

4. Let $u(x,t)$ be defined for $0 \leq x \leq 1$ and $t > 0$, and consider the following PDE

$$u_{tt} + 2\beta u_t = u_{xx}, \qquad u(0,t) = u(1,t) = 0, \quad u(x,0) = x(1-x), \quad u_t(x,0) = 1.$$

where $0 < \beta < 1$ is a constant. This is the wave equation, where the transverse vibrations take place in a medium that imparts a resistance proportional to the instantaneous velocity.

   (a) Describe and sketch this situation at $t = 0$.

   (b) Assume that there is a solution of the form $u(x,t) = f(x)g(t)$. Plug this back into the PDE and get an ODE for $g(t)$ and a Sturm-Liouville problem for $f(x)$.

   (c) The Sturm-Liouville problem should be familiar: $f'' = -\lambda f$, $f(0) = f(1) = 0$, and we've seen that the eigenvalues are $\lambda_n = n^2$ for $n = 1, 2, \ldots$ and the eigenfunctions are $f_n(x) = b_n \sin(n\pi x)$. The equation for $g(t)$ should be $g'' + 2\beta g' + n^2 g = 0$. Solve this ODE for $g(t)$.

   (d) Write down the general solution to this PDE.

   (e) Use the initial conditions to find the particular solution solving the boundary and initial conditions. (Feel free to use WolframAlpha to compute any derivatives you may need.)

   (f) What is the long-term behavior of this system? Give both a mathematical and physical justification.

5. In quantum mechanics, the function $\psi(x,t)$ represents the wave function of a quantum system. One can interpret this as a probability density on th epossible quantum states.

   (a) Solve the following initial boundary value problem to Schrödiner's equation:

$$i\hbar\psi_t = -\psi_{xx}, \qquad \psi(t,0) = \psi(t,1), \qquad \psi(0,x) = 1.$$

   The Dirchlet boundary conditions represent the case when the particle cannot escape from the interval $[0,1]$. (Note the difference from what this represents in the heat equation!)

   (b) Solve the same initial boundary value problem, but with Neumann boundary conditions:

$$i\hbar\psi_t = -\psi_{xx}, \qquad \psi_x(t,0) = \psi_x(t,1), \qquad \psi(0,x) = 1.$$