

TOPICS: PDES IN OTHER COORDINATE SYSTEMS

In the process of solving these problems, you will encounter several ODEs, Sturm-Liouville problems, PDEs, and Fourier series, many of which you have encountered before. You do not need to re-derive the solutions of anything you have previously solved.

1. Let $u(r, \theta)$ be a function defined on the disk of radius R . Consider the following boundary value problem for *Laplace's equation* in polar coordinates:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad u(r, \theta + 2\pi) = u(r, \theta), \quad u(R, \theta) = 2 - 3\cos\theta + 5\sin 2\theta.$$

- (a) Assume that a solution has the form $u(r, \theta) = R(r)T(\theta)$. Plug this back in and separate variables to get an equation for R and T , including boundary conditions.
 - (b) Solve the ODEs for $R(r)$ and $T(\theta)$, and determine all possible eigenvalues λ_n . Make sure to impose the additional requirement that $R(0)$ exists.
 - (c) Find the general solution to Laplace's equation in polar coordinates.
 - (d) Plug in $r = R$ to find the particular solution to this boundary value problem.
2. Let $u(r, \theta, t)$ be a function defined on the disk of radius $R = 1$, and for all $t \geq 0$. Consider the following initial/boundary value problem for the *heat equation* in polar coordinates:

$$\begin{aligned} u_t &= c^2 \Delta u, & u(r, \theta + 2\pi) &= u(r, \theta), & u(r, \theta, 0) &= 1 - r^2 + h(r, \theta) \\ u(1, \theta, t) &= 2 - 3\cos\theta + 5\sin 2\theta. \end{aligned}$$

Here, $h(r, \theta)$ denotes the *steady-state* solution.

- (a) What is $h(r, \theta)$?
 - (b) Make the change of variables $v(r, \theta, t) = u(r, \theta, t) - h(r, \theta)$, and re-write the PDE above, including the boundary and initial conditions, in terms of v instead of u .
 - (c) Find the general solution for this *homogeneous* PDE using separation of variables. Assume that $v(r, \theta, t) = f(r, \theta)g(t)$.
 - (d) Find the particular solution that satisfies the initial condition.
3. Let $u(r, \theta, t)$ be a function defined on the disk of radius $R = 1$, and for all $t \geq 0$. Consider the following initial/boundary value problem for the *wave equation* in polar coordinates:

$$\begin{aligned} u_{tt} &= c^2 \Delta u, & u(r, \theta + 2\pi) &= u(r, \theta) & u(r, \theta, 0) &= 1 - r^2 \\ & & u(1, \theta, t) &= 0, & u_t(r, \theta, 0) &= 0. \end{aligned}$$

- (a) Briefly describe, and sketch, a physical situation which this models. Be sure to explain the effect of the boundary conditions and both initial conditions. [*Hint*: What does the function $u(r, \theta, 0) = 1 - r^2$ look like?]
- (b) Assume that there is a solution of the form $u(r, \theta, t) = f(r, \theta)g(t)$. Plug this back in and separate variables to get an ODE for g and a PDE for f . Include boundary conditions for f , and one initial condition for g .

- (c) Find the general solution to this BVP.
- (d) Find the particular solution that additionally satisfies the initial conditions.
4. Let $u(r, \theta, \phi, t)$ be the temperature of a sphere of radius $R = \pi$. Assume that the initial temperature is constant, and that temperature does not depend on the latitude or longitude. In this case, $u(r, \theta, \phi, t) = u(r, t)$, and the heat equation reduces down to the following:

$$u_t = c^2(u_{rr} + \frac{2}{r}u_r), \quad u(\pi, t) = 0, \quad u(r, 0) = T_0.$$

There is an implied boundary condition at $r = 0$, that $u(0, t)$ is finite.

- (a) Assume that there is a solution of the form $u(r, t) = f(r)g(t)$. Separate variables to get two equations, and ODE for g , and a (singular) Sturm-Liouville problem for f .
- (b) Solve the Sturm-Liouville problem for f . [*Hint*: One way would be to use the power series method. However, a much easier way is to define $y(r) = rf(r)$, and re-write the problem in terms of y .]
- (c) Write the general solution to this PDE.
- (d) Find the particular solution that additionally satisfies the initial condition. Leave the formulas for coefficients in terms of integrals; no need to solve them.
5. Consider a sphere of radius $R = 1$, and suppose that $u(r, \phi)$ represents a potential that depends only on the radius $r \in [0, 1]$ and latitude $\phi \in [0, \pi]$. In this problem, we will solve Laplace's equation under these conditions. The boundary value problem is

$$u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2 \sin \phi}(\sin \phi u_\phi)_\phi = 0, \quad u(1, \phi) = f(\phi).$$

- (a) Assume $u(r, \phi) = R(r)Y(\phi)$, and derive the following two equations

$$1((1 - x^2)y')' = \lambda y, \quad (r^2 R')' = \lambda R,$$

where $x = \cos \phi$ for $-1 < x < 1$ and $y(x) = Y(\cos^{-1}(x))$.

- (b) The equation for $y(x)$ may look familiar – it is *Legendre's equation* (see HW 4, 6, and 9). Recall that it has bounded, continuous solutions on $[-1, 1]$ when

$$\lambda_n = n(n+1), \quad y_n(x) = P_n(x), \quad n = 0, 1, 2, \dots$$

These are the eigenvalues and eigenfunctions of the related (singular) Sturm-Liouville problem (see HW 9). Carry out the details of deriving the general solution to Laplace's equation, which will be

$$u(r, \phi) = \sum_{n=0}^{\infty} c_n r^n P_n(\cos \phi), \quad \text{where} \quad c_n = \frac{1}{||P_n||^2} \int_0^\pi f(\phi) P_n(\cos \phi) \sin \phi \, d\phi.$$

- (c) If $f(\phi) = \sin \phi$, find an approximate, *fourth-order* solution. That is, truncate it after the $n = 4$ term. *Hint*: The Legendre polynomials can be derived from the following formula:

$$P_n(x) = \frac{1}{n!2^n} \frac{d^n}{dx^n} [(x^2 - 1)^n].$$