Lecture 1.1: Vector spaces

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Math 4340, Advanced Engineering Mathematics

Motivation

A (real-valued) function f is linear if

$$f(ax+by) = af(x) + bf(y).$$

In other words, if you can "break apart sums and pull out constants".

Many common structures and operations have this property. For example:

- derivatives: $\frac{d}{dx}(au+bv) = a\frac{du}{dx} + b\frac{dv}{dx}$
- integrals: $\int (au+bv)\,dx = a\int u\,dx + b\int v\,dx$
- matrices and vectors: $\mathbf{M}(a\mathbf{x}+b\mathbf{y}) = a\mathbf{M}\mathbf{x} + b\mathbf{M}\mathbf{y}$
- Laplace transforms: $\mathcal{L}(af+bg) = a\mathcal{L}(f) + b\mathcal{L}(g)$
- Solutions of certain ODEs: If y_1 and y_2 solve $y'' + k^2y = 0$, then so does $C_1y_1 + C_2y_2$.

We encounter this type of linear structure *all the time* without realizing it.

A beginning linear algebra class usually focuses on systems of equations and matrix algebra.

An $m \times n$ matrix encodes a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$. Elements in these sets are "vectors".

But this is just a special case of the "bigger picture". We'll begin this course by peeking at this structure, which underlies nearly every aspect of the mathematics in this class.

Vector spaces

Definition

A vector space consists of a set V (of "vectors") and a set $\mathbb{F}$ (of "scalars"; usually $\mathbb{R}$ or $\mathbb{C}$) that is:

- closed under addition: $v, w \in V \implies v + w \in V$
- closed under scalar multiplication: $v \in V$, $c \in \mathbb{F} \implies cv \in V$

Remark

We can deduce some easy consequences:

- $\mathbf{0} \in V$
- $v \in V \implies -v \in V$

If $\mathbb{F} = \mathbb{R}$, we say V is a "real vector space", an "$\mathbb{R}$-vector space", or a "vector space over $\mathbb{R}$".

A "complex vector space" is defined similarly (i.e., if $\mathbb{F} = \mathbb{C}$).

Blanket assumption

Unless specified otherwise, we will assume by default that $\mathbb{F} = \mathbb{R}$.

Vector spaces

Examples

1. $V = \mathbb{R}^n = \big\{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R}\big\}$.

   "+": $(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n) \in \mathbb{R}^n$

   "$\cdot$": $c \cdot (x_1, \ldots, x_n) = (cx_1, \ldots, cx_n) \in \mathbb{R}^n$
2. $V = \mathbb{C}^n = \big\{(z_1, \ldots, z_n) \mid z_i \in \mathbb{C}\big\}$.
3. $V = \mathbb{R}_n[x] = \{a_0 + a_1x + \cdots + a_nx^n \mid a_i \in \mathbb{R}\}$. "polynomials of degree $\leq n$"
4. $V = \mathbb{R}[x] = \{a_0 + a_1x + \cdots + a_kx^k \mid a_i \in \mathbb{R}\}$. "polynomials of arbitrary degree"
5. $V = \mathbb{R}[[x]] = \{a_0 + a_1x + a_2x^2 + \cdots \mid a_i \in \mathbb{R}\}$. "power series"
6. $V = \mathcal{C}^1(\mathbb{R}) =$ (once) differentiable real-valued functions s.t. $f'(x)$ is continuous.
7. $V = \mathcal{C}^\infty(\mathbb{R}) =$ infinitely differentiable functions; $f^{(k)}(x)$ continuous for all k.
8. $V = \mathrm{Per}_{2\pi} =$ piecewise continuous functions with $f(x) = f(x + 2\pi)$, i.e., period $T = 2\pi/n$ for some $n \in \mathbb{N}$.

Non-examples

1. Polynomials with degree n. [e.g., $(x^n + 1) + (2 - x^n) = 3$]
2. The upper half-plane in $\mathbb{R}^2$. [e.g., $-1 \cdot (0, 1) = (0, -1)$]
3. A line (or plane) not through the origin. [e.g., $0 \cdot v = \mathbf{0}$]

Subspaces

Definition

If V is a vector space (over $\mathbb{F}$), then a subspace is a subset $W \subseteq V$ that is also a vector space (over $\mathbb{F}$). We write $W \leq V$.

Examples

1. V and $\{\mathbf{0}\}$ are always subspaces of V.
2. Let $V = \{(x, y, z) \mid x, y, z \in \mathbb{R}\} = \mathbb{R}^3$ and $W = \{(x, y, 0) \mid x, y \in \mathbb{R}\} \cong \mathbb{R}^2$.
   Then W is a subspace of V.
3. Clearly, $\mathbb{R}_n[x] \subsetneq \mathbb{R}[x] \subsetneq \mathbb{R}[[x]]$ as subsets.
   - $\mathbb{R}_n[x]$ is a subspace of $\mathbb{R}[x]$ and $\mathbb{R}[[x]]$.
   - $\mathbb{R}[x]$ is a subspace of $\mathbb{R}[[x]]$.
4. $\mathcal{C}^\infty(\mathbb{R})$ is a subspace of $\mathcal{C}^1(\mathbb{R})$. Also, note that

$$\mathcal{C}^1(\mathbb{R}) \supsetneq \mathcal{C}^2(\mathbb{R}) \supsetneq \mathcal{C}^3(\mathbb{R}) \supsetneq \cdots \supsetneq \mathcal{C}^\infty(\mathbb{R}).$$

Remark

Subspaces in $\mathbb{R}^n$ "look like" hyperplanes (lines, planes, etc.) through the origin.

Subspaces

Definition

If V is a vector space (over $\mathbb{F}$), then a subspace is a subset $W \subseteq V$ that is also a vector space (over $\mathbb{F}$). We write $W \leq V$.

Non-examples

1. The unit circle in $\mathbb{R}^2$ $\quad (\subseteq \mathbb{R}^2)$
2. Polynomials of degree n $\quad (\subseteq \mathbb{R}_n[x])$
3. Upper half-plane $\quad (\subseteq \mathbb{R}^2)$
4. The line $y = 2x + 3$ $\quad (\subseteq \mathbb{R}^2)$
5. The plane $\{(x, y, 1) \mid x, y \in \mathbb{R}\}$ $\quad (\subseteq \mathbb{R}^3)$
6. Piecewise continuous functions with period *exactly* 2π $\quad (\subseteq \mathrm{Per}_{2\pi})$

How to determine whether W is a subspace of V

Given a collection of "vectors" $W \subseteq V$, ask:

- Is it closed under addition?
- Is it closed under scalar multiplication?