Lecture 1.3: Linear maps

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Linear maps

Definition

A linear map is a function $T\colon V \to W$ between vector spaces V and W satisfying

$$T(ax + by) = aT(x) + bT(y), \qquad \text{for all } x, y \in V;\ \ a, b \in \mathbb{F}.$$

When the vector spaces consist of functions (e.g., $\mathcal{C}^\infty(\mathbb{R})$, $\mathbb{R}[x]$, or $\mathrm{Per}_{2\pi}(\mathbb{R})$), we often use the term linear operator.

For example, $\dfrac{d}{dx}$ and $\displaystyle\int$ are linear operators.

When our vector space is $\mathbb{R}^n$, we usually just say linear "map" or "function".

For example, $f(x) = 3x$ and $f(x_1, x_2) = 8x_1 - 3x_2$ are linear functions.

Definition

The kernel (or nullspace) of a linear map $T\colon V \to W$, denoted $\ker(T)$ is the set of vectors such that $T(v) = 0$:

$$\ker(T) = \big\{v \in V \mid T(v) = 0\big\}.$$

The image (or range) of T is the set $T(V)$, i.e.,

$$\mathrm{im}(T) = \big\{T(v) \mid v \in V\big\}.$$

Examples

From calculus

Let $V = W = \mathcal{C}^\infty(\mathbb{R})$.

1. $T = \frac{d}{dx}$ is a linear operator:
$$T\colon f(x) \longmapsto f'(x).$$

2. $T = \int$ is a linear operator:
$$T\colon f(x) \longmapsto \int f(x)\,dx.$$

3. The Laplace transform $\mathcal{L}$ is a linear operator:
$$\mathcal{L}\colon f(t) \longmapsto \int_0^\infty f(t)\,e^{-st}\,dt.$$

Examples

Matrices

Any 2×2 matrix is a linear map $A\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix}.$$

Facts

- $|\det A| =$ scaling factor (=area of parallelogram); negative denotes reflection.
- A is invertible iff $\det A \neq 0$.
- In general, an $m \times n$ matrix is a linear map $A\colon \mathbb{R}^n \to \mathbb{R}^m$
- $\operatorname{im} A$ and $\ker A$ are both subspaces, and they satisfy

$$\dim(\operatorname{im} A) + \dim(\ker A) = n.$$

  Intuitively, every "dimension" either gets collapsed or persists.

Connections between linear operators and ODEs

Preview!

Let $V = W = \mathcal{C}^\infty(\mathbb{R})$.

- $T = \dfrac{d^2}{dx^2}$ is a linear operator: $y \longmapsto y''$.

$$\ker(T) = \big\{y(x) \mid y''(x) = 0\big\} = \big\{C_1 x + C_2 \mid C_1, C_2 \in \mathbb{R}\big\}.$$

- $T = \dfrac{d^2}{dt^2} + k^2$ is a linear operator: $y \longmapsto y'' + k^2 y$.

$$\ker(T) = \big\{y(t) \mid y'' + k^2 y = 0\big\} = \big\{C_1 \cos kt + C_2 \sin kt \mid C_1, C_2 \in \mathbb{R}\big\}.$$

- $T = \dfrac{d^2}{dt^2} + t^2$ is a linear operator: $y \longmapsto y'' + t^2 y$.

$$\ker(T) = \big\{y(t) \mid y'' + t^2 y = 0\big\}.$$

Connections between linear operators and ODEs

Big idea

The kernel (or nullspace) of these linear differential operators are solutions to linear homogeneous differential equations.

Since $\ker(T)$ is a vector space, the set of solutions (i.e., the general solution) to a linear homogeneous ODE is a vector space.