Lecture 2.2: Linear independence and the Wronskian

Matthew Macauley

Department of Mathematical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

Math 4340, Advanced Engineering Mathematics

Motivation

Question

To solve an n^{th} order linear homogeneous ODE, we need to (somehow) find n linearly independent solutions, i.e., a basis for the solution space.

Given n functions $y_1, \dots, y_n$, how can we determine if they are linearly independent?

The case $n = 1$ is easy: $\{y_1\}$ is independent iff $y_1 \not\equiv 0$.

The case $n = 2$ isn't difficult, but we have to be careful: $\{y_1, y_2\}$ is independent iff $y_1 \neq cy_2$.

Example

Are the functions $y_1(t) = \sin 2t$ and $y_2(t) = \sin t \cos t$ linearly independent?

Things can get much more complicated if $n > 2$.

Example

Are the functions $y_1(t) = e^{2t}$, $y_2(t) = e^{-2t}$, and $y_3(t) = \cosh 2t$ linearly independent?

What about $y_1(t) = e^{2t} - 5e^{-2t}$, $y_2(t) = 3e^{-2t} - \cosh 2t$, and $y_3(t) = e^{2t} + e^{-2t} + \cosh 2t$?

The Wronskian

Definition

Let $f_1, \ldots, f_n$ be functions defined on an interval $[\alpha, \beta]$. Their Wronskian is defined by

$$W(f_1, \ldots, f_n)(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

Example

Compute the Wronskian of $y_1(t) = \cos t$ and $y_2(t) = \sin t$.

Theorem

If $W(f_1, \ldots, f_n) \not\equiv 0$, then $\{f_1, \ldots, f_n\}$ is linearly independent.

Remark

Unfortunately, $W(f_1, \ldots, f_n) \equiv 0$ does *not* necessarily imply that $\{f_1, \ldots, f_n\}$ is linearly independent.

Example: $f_1(x) = x^2$ and $f_2(x) = |x| \cdot x$.

Key property of the Wronskian

Abel's theorem

Let y_1, y_2 be solutions to $y'' + a(t)y' + b(t)y = 0$, where a, b are continuous in $[\alpha, \beta]$. Then the Wronskian is

$$W(y_1, y_2)(t) = Ce^{-\int a(t)\,dt},$$

for some constant C. Moreover, $W(t)$ is either identically 0, or never zero, on $[\alpha, \beta]$.

Proof

Claim: The Wronskian $W(y_1, y_2)$ is a solution to the first order ODE $W' + a(t)W = 0$.

Reduction of order

Abel's theorem

Let y_1, y_2 be solutions to $y'' + a(t)y' + b(t)y = 0$, where a, b are continuous in $[\alpha, \beta]$. Then the Wronskian is

$$W(y_1, y_2)(t) = Ce^{-\int a(t)\,dt},$$

for some constant C. Moreover, $W(t)$ is either identically 0, or never zero, on $[\alpha, \beta]$.

Corollary

If we only know one solution $y_1(t)$ of an 2nd order ODE, then we can solve for the other:

$$y_2' + a(t)y_2 = g(t) : \qquad y_2' - \frac{y_1'}{y_1}y_2 = W(y_1, y_2)/y_1 = e^{-\int a(t)\,dt}/y_1.$$

Proof

Reduction of order

Example

Solve $y'' + 4y' + 4y = 0$.

More applications of Abel's theorem

Abel's theorem

Let y_1, y_2 be solutions to $y'' + a(t)y' + b(t)y = 0$, where a, b are continuous in $[\alpha, \beta]$. Then the Wronskian is

$$W(y_1, y_2)(t) = Ce^{-\int a(t)\,dt},$$

for some constant C. Moreover, $W(t)$ is either identically 0, or never zero, on $[\alpha, \beta]$.

Example

Compute the Wronskian of $y_1(t) = \sin 2t$ and $y_2(t) = \sin t \cos t$.

More applications of Abel's theorem

Abel's theorem

Let y_1, y_2 be solutions to $y'' + a(t)y' + b(t)y = 0$, where a, b are continuous in $[\alpha, \beta]$. Then the Wronskian is

$$W(y_1, y_2)(t) = Ce^{-\int a(t)\,dt},$$

for some constant C. Moreover, $W(t)$ is either identically 0, or never zero, on $[\alpha, \beta]$.

Example

There is no ODE of the form $y'' + a(t)y' + b(t)y = 0$ that has both $y_1(t) = t^2$ and $y_2(t) = t + 1$ as solutions.