# Lecture 3.2: Computing Fourier series and exploiting symmetry

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Math 4340, Advanced Engineering Mathematics

# Exploiting symmetry

There are many shortcuts to computing Fourier series: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}$.

## Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is

- even if $f(x) = f(-x)$ for all $x \in \mathbb{R}$,
- odd if $f(x) = -f(-x)$ for all $x \in \mathbb{R}$.

| even | odd | neither |
|---|---|---|
| $x^n$ (even $n$) | $x^n$ (odd $n$) | $x^2 + x^3$. |
| $\cos nx$ | $\sin nx$ | $e^{inx}$ $(= \cos nx + i \sin nx)$ |
| symmetric about $y$-axis | symmetric about origin | neither |

## Why we care

- If $f$ is even, then $\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx$.
- If $f$ is odd, then $\int_{-L}^{L} f(x)\,dx = 0$.

# Exploiting symmetry: $f(x) = \frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$

## Big shortcut

If $f$ is even, then every $b_n = 0$:

$$b_n = \langle f, \sin\tfrac{n\pi x}{L}\rangle = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\sin\tfrac{n\pi x}{L}}_{\text{even}\,\cdot\,\text{odd}\,=\,\text{odd}}\,dx = 0.$$

If $f$ is odd, then every $a_n = 0$:

$$a_n = \langle f, \cos\tfrac{n\pi x}{L}\rangle = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\cos\tfrac{n\pi x}{L}}_{\text{odd}\,\cdot\,\text{even}\,=\,\text{odd}}\,dx = 0.$$

## Small shortcut

If $f$ is even, then

$$a_n = \langle f, \cos\tfrac{n\pi x}{L}\rangle = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\cos\tfrac{n\pi x}{L}}_{\text{even}\,\cdot\,\text{even}\,=\,\text{even}}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\tfrac{n\pi x}{L}\,dx.$$

If $f$ is odd, then

$$b_n = \langle f, \sin\tfrac{n\pi x}{L}\rangle = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\sin\tfrac{n\pi x}{L}}_{\text{odd}\,\cdot\,\text{odd}\,=\,\text{even}}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\tfrac{n\pi x}{L}\,dx.$$

# An odd square wave

## Example 1

Consider the square wave of period 2 defined by $f(x) = \begin{cases} 1 & 0 < x < 1 \\ -1 & -1 < x < 0 \end{cases}$

# A sawtooth wave

## Example 2

Consider the sawtooth wave defined by $f(x) = x$ on $(-L, L)$ and extended to be periodic.

# An even function

## Example 3

Consider the function defined by $f(x) = x^2$ on $[-1, 1]$ and extended to be periodic.

# The average value of a Fourier series

## Proposition

For any Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \tfrac{n\pi x}{L} + b_n \sin \tfrac{n\pi x}{L},$$

the average value of $f(x)$ is $\dfrac{a_0}{2}$.

## Exercise

Consider a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

What is the Fourier series of the function obtained by

(i) reflecting $f$ across the $y$-axis?

(ii) reflecting $f$ across the $x$-axis?

(iii) reflecting $f$ across the origin?