Lecture 3.5: Complex inner products and Fourier series

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Review of complex numbers

Euler's formula

- $e^{i\theta} = \cos\theta + i\sin\theta$
- $e^{-i\theta} = \cos\theta - i\sin\theta$
- $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$
- $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

Real vs. complex vector spaces

Until now, we have primarily been dealing with $\mathbb{R}$-vector spaces. Things are a little different with $\mathbb{C}$-vector spaces.

To understand why, compare the notion of *norm* for real vs. complex numbers.

- For any real number $x \in \mathbb{R}$, its norm (distance from $\mathbf{0}$) is $|x| = \sqrt{x^2} \in \mathbb{R}$.
- For any complex number $z = a + bi \in \mathbb{C}$, its norm (distance from $\mathbf{0}$) is defined by

$$|z| := \sqrt{z\overline{z}} = \sqrt{(a+bi)(a-bi)} = \sqrt{a^2+b^2}.$$

Let's now go from $\mathbb{R}$ and $\mathbb{C}$ to $\mathbb{R}^2$ and $\mathbb{C}^2$.

- For any vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$, its norm (distance from $\mathbf{0}$) is

$$||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{\mathbf{v}^T\mathbf{v}} = \sqrt{v_1^2 + v_2^2}.$$

- For any $\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathbb{C}^2$, with $z_1 = a + bi$, $z_2 = c + di$, its norm is defined by

$$||\mathbf{z}|| := \sqrt{\overline{\mathbf{z}}^T\mathbf{z}} = \sqrt{|z_1|^2 + |z_2|^2}.$$

For example, let's compute the norms of $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathbb{R}^2$ and $\mathbf{v} = \begin{bmatrix} i \\ i \end{bmatrix} \in \mathbb{C}^2$.

Real vs. complex inner products

Big idea

The norm in an $\mathbb{R}$-vector space is defined using a real inner product, e.g.,

$$\mathbf{v}\cdot\mathbf{w} := \mathbf{w}^T\mathbf{v} = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix}\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \sum v_i w_i.$$

The norm in a $\mathbb{C}$-vector space is defined using a complex inner product, e.g.,

$$\mathbf{z}\cdot\mathbf{w} := \overline{\mathbf{w}}^T\mathbf{z} = \begin{bmatrix} \overline{w_1} & \cdots & \overline{w_n} \end{bmatrix}\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \sum z_i \overline{w_i}.$$

Definition

Let V be an $\mathbb{C}$-vector space. A function $\langle -,- \rangle \colon V \times V \to \mathbb{C}$ is a (complex) inner product if it satisfies (for all $\mathbf{u},\mathbf{v},\mathbf{w},\in V$, $c\in\mathbb{C}$):

(i) $\langle \mathbf{u}+\mathbf{v},\mathbf{w}\rangle = \langle \mathbf{u},\mathbf{v}\rangle + \langle \mathbf{v},\mathbf{w}\rangle$

(ii) $\langle c\mathbf{v},\mathbf{w}\rangle = c\langle \mathbf{v},\mathbf{w}\rangle$ and $\langle \mathbf{v},c\mathbf{w}\rangle = \overline{c}\langle \mathbf{v},\mathbf{w}\rangle$

(iii) $\langle \mathbf{v},\mathbf{w}\rangle = \overline{\langle \mathbf{w},\mathbf{v}\rangle}$

(iv) $\langle \mathbf{v},\mathbf{v}\rangle \geq 0$, with equaility if and only if $\mathbf{v}=0$.

Periodic functions as a $\mathbb{C}$-vector space

Recall the $\mathbb{R}$-vector space (we'll abuse terminology and allow infinite sums)

$$\mathrm{Per}_{2\pi}(\mathbb{R}) := \mathrm{Span}\Big\{\{1,\, \cos x,\, \cos 2x, \dots\} \cup \{\sin x,\, \sin 2x, \dots\}\Big\}$$

Let $\mathrm{Per}_{2\pi}(\mathbb{C})$ denote the same vector space but with coefficients from $\mathbb{C}$. Since

$$\cos nx = \frac{e^{inx} + e^{-inx}}{2}, \qquad \sin nx = \frac{e^{inx} - e^{-inx}}{2i},$$

a better way to write $\mathrm{Per}_{2\pi}(\mathbb{C})$ is using a different basis:

$$\mathrm{Per}_{2\pi}(\mathbb{C}) := \mathrm{Span}\left\{\dots, e^{-2ix},\, e^{-ix},\, 1,\, e^{ix},\, e^{2ix}, \dots\right\}$$

It turns out that this basis is orthonormal with respect to the inner product

$$\langle f, g\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx\,.$$

It is quite easy to verify this:

$$\left\langle e^{inx},\, e^{imx}\right\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\,\overline{e^{imx}}\, dx = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

Formulas for the Fourier coefficients

Definition / Theorem

If $f(x)$ is a piecewise continuous $2L$-periodic function, then its complex Fourier series is

$$f(x)=\sum_{n=-\infty}^{\infty} c_n e^{\frac{i\pi nx}{L}} = c_0+\sum_{n=1}^{\infty}\left(c_n e^{\frac{i\pi nx}{L}}+c_{-n}e^{-\frac{i\pi nx}{L}}\right)$$

where the complex Fourier coefficients are

$$c_0=\langle f,\,1\rangle=\frac{1}{2L}\int_{-L}^{L} f(x)\,dx,\qquad c_n=\left\langle f,\,e^{\frac{i\pi nx}{L}}\right\rangle=\frac{1}{2L}\int_{-L}^{L} f(x)e^{-\frac{i\pi nx}{L}}\,dx.$$

Computations

Example 1: square wave

Find the complex Fourier series of $f(x) = \begin{cases} 1, & 0 < x < \pi \\ -1, & \pi < x < 2\pi. \end{cases}$

Computations

Example 2

Compute the complex Fourier series of the 2π-periodic extension of the function e^x defined on $-\pi < 0 < \pi$.