Lecture 4.3: Self-adjoint linear operators

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Why self-adjoint operators are nice

Definition

Let V be a vector space with inner product $\langle -, - \rangle$. A linear operator $L \colon V \to V$ is self-adjoint if

$$\langle Lf, g \rangle = \langle f, Lg \rangle, \qquad \text{for all } f, g \in V.$$

Theorem

If L is a self-adjoint linear operator, then:

(i) All eigenvalues of L are real.

(ii) Eigenfunctions corresponding to distinct eigenvalues are orthogonal.

Proof

A one-variable example

Remark

The linear operator $L = \frac{d^2}{dx^2} = \partial_x^2$ on the space $\mathcal{C}^\infty[0,1]$ is *not* self-adjoint.

Dirichlet vs. Neumann boundary conditions

Proposition

The linear operator $L = \dfrac{d^2}{dx^2} = \partial_x^2$ on either of the subspaces

- $\mathcal{C}_0^\infty[a,b] := \Big\{ f \in \mathcal{C}^\infty[a,b] : f(a) = f(b) = 0 \Big\}$
- $\mathcal{C}_\perp^\infty[a,b] := \Big\{ f \in \mathcal{C}^\infty[a,b] : f'(a) = f'(b) = 0 \Big\}$

is self-adjoint.

Mixed boundary conditions

Proposition

The linear operator $L = \dfrac{d^2}{dx^2} = \partial_x^2$ on the subspace

$$\mathcal{C}^\infty_{\alpha,\beta}[a,b] := \Big\{ f \in \mathcal{C}^\infty[a,b] :\ \alpha_1 f(a) + \alpha_2 f'(a) = 0, \quad \beta_1 f(b) + \beta_2 f'(b) = 0 \Big\},$$

where $\alpha_1^2 + \alpha_2^2 > 0$ and $\beta_1^2 + \beta_2^2 > 0$, is self-adjoint.

A multivariate example

Theorem

Let $R \subset \mathbb{R}^n$ be a bounded region with a smooth boundary B. Then the Laplacian operator

$$\Delta = \nabla^2 := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} = \sum_{i=1}^{n} \partial_{x_i}^2$$

is self-adjoint on the space $V = \mathcal{C}_0^{\infty}(R)$ of infinitely differentiable functions that vanish on B.

The eigenfunctions of ∇^2 are solutions to the PDE

$$\nabla^2 f = \lambda f,$$

called the Helmholtz equation.

An example from quantum mechanics

Definition

The Hamiltonian is a self-adjoint operator, defined by

$$H = -\frac{\hbar}{2m}\nabla^2 + V.$$

This describes the energy of a particle of mass m in a real potential field V.

The eigenfunctions ψ of H represent the stationary quantum states, and the eigenvalues E describe the energy levels of these states. They are solutions to the following PDE, called Schrödinger equation:

$$H\psi = E\psi.$$