Lecture 4.6: Some special orthogonal functions

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Motivation

Recall that every 2nd order linear homogeneous ODE, $y'' + P(x)y' + Q(x)y = 0$ can be written in self-adjoint or "Sturm-Liouville form":

$$-\frac{d}{dx}\Big(p(x)y'\Big) + q(x)y = \lambda w(x)y, \qquad \text{where } p(x),\ q(x),\ w(x) > 0.$$

Many of these ODEs require the Frobenius method to solve.

Examples from physics and engineering

- Legendre's equation: $(1-x^2)y'' - 2xy' + n(n+1)y = 0$. *Used for modeling spherically symmetric potentials in the theory of Newtonian gravitation and in electricity & magnetism (e.g., the wave equation for an electron in a hydrogen atom).*
- Parametric Bessel's equation: $x^2y'' + xy' + (\lambda x^2 - \nu^2)y = 0$. *Used for analyzing vibrations of a circular drum.*
- Chebyshev's equation: $(1-x^2)y'' - xy' + n^2y = 0$. *Arises in numerical analysis techniques.*
- Hermite's equation: $y'' - 2xy' + 2ny = 0$. *Used for modeling simple harmonic oscillators in quantum mechanics.*
- Laguerre's equation: $xy'' + (1-x)y' + ny = 0$. *Arises in a number of equations from quantum mechanics.*
- Airy's equation: $y'' - k^2xy = 0$. *Models the refraction of light.*

Legendre's differential equation

Consider the following Sturm-Liouville problem, defined on $(-1, 1)$:

$$-\frac{d}{dx}\Big[(1-x^2)\frac{d}{dx}y\Big] = \lambda y, \qquad \Big[p(x) = 1-x^2, \quad q(x) = 0, \quad w(x) = 1\Big].$$

The eigenvalues are $\lambda_n = n(n+1)$ for $n = 1, 2, \dots$, and the eigenfunctions solve Legendre's equation:

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0.$$

For each n, one solution is a degree-n "Legendre polynomial"

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\big[(x^2-1)^n\big].$$

They are orthogonal with respect to the inner product $\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)g(x)\,dx$.

It can be checked that

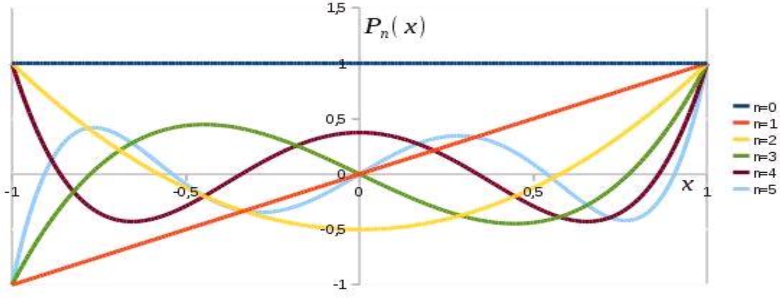
$$\langle P_m, P_n\rangle = \int_{-1}^{1} P_m(x)P_n(x)\,dx = \frac{2}{2n+1}\delta_{mn}.$$

By orthogonality, every function f, continuous on $-1 < x < 1$, can be expressed using Legendre polynomials:

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x), \qquad \text{where} \quad c_n = \frac{\langle f,\ P_n\rangle}{\langle P_n,\ P_n\rangle} = (n+\tfrac{1}{2})\,\langle f,\ P_n\rangle$$

Legendre polynomials

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$
$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$
$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$
$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$
$$P_6(x) = \frac{1}{8}(231x^6 - 315x^4 + 105x^2 - 5)$$
$$P_7(x) = \frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x)$$



Parametric Bessel's differential equation

Consider the following Sturm-Liouville problem on $[0, a]$:

$$-\frac{d}{dx}(xy') - \frac{\nu^2}{x}y = \lambda xy, \qquad \Big[p(x) = x, \quad q(x) = -\frac{\nu^2}{x}, \quad w(x) = x\Big].$$

For a fixed ν, the eigenvalues are $\lambda_n = \omega_n^2 := \alpha_n^2/a^2$, for $n = 1, 2, \ldots$.

Here, α_n is the n^{th} positive root of $J_\nu(x)$, the Bessel functions of the first kind of order ν.

The eigenfunctions solve the parametric Bessel's equation:

$$x^2y'' + xy' + (\lambda x^2 - \nu^2)y = 0.$$

Fixing ν, for each n there is a solution $J_{\nu n}(x) := J_\nu(\omega_n x)$.

They are orthogonal with repect to the inner product $\langle f, g\rangle = \int_0^a f(x)g(x)\,x\,dx$.

It can be checked that

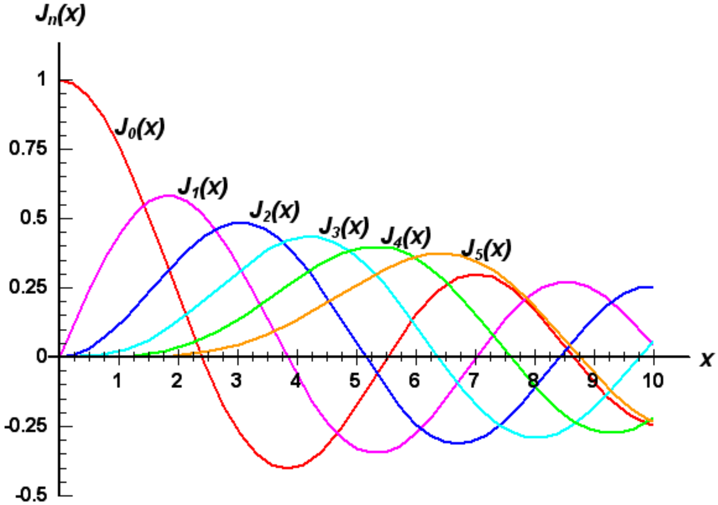
$$\langle J_{\nu n}, J_{\nu m}\rangle = \int_0^a J_\nu(\omega_n x)J_\nu(\omega_m x)\,x\,dx = 0, \quad \text{if } n \neq m.$$

By orthogonality, every continuous function $f(x)$ on $[0, a]$ can be expressed in a "Fourier-Bessel" series:

$$f(x) \sim \sum_{n=0}^{\infty} c_n J_\nu(\omega_n x), \qquad \text{where} \quad c_n = \frac{\langle f,\, J_{\nu n}\rangle}{\langle J_{\nu n},\, J_{\nu n}\rangle}.$$

Bessel functions (of the first kind)

$$J_\nu(x) = \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!(\nu+m)!} \left(\frac{x}{2}\right)^{2m+\nu} .$$



Fourier-Bessel series from $J_0(x)$

$$f(x) \sim \sum_{n=0}^{\infty} c_n J_0(\omega_n x), \qquad J_0(x) = \sum_{m=0}^{\infty} (-1)^m \frac{1}{(m!)^2} \left(\frac{x}{2}\right)^{2m}$$

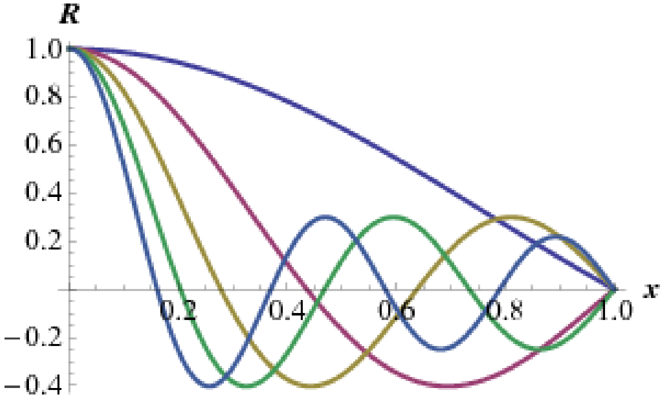


Figure: First 5 solutions to $(xy')' = -\lambda x^2$.

Fourier-Bessel series from $J_3(x)$

The Fourier-Bessel series using $J_3(x)$ of the function $f(x) = \begin{cases} x^3 & 0 < x < 10 \\ 0 & x > 10 \end{cases}$ is

$$f(x) \sim \sum_{n=0}^{\infty} c_n J_3(\omega_n x/10), \qquad J_3(x) = \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!(3+m)!} \left(\frac{x}{2}\right)^{2m+3}.$$

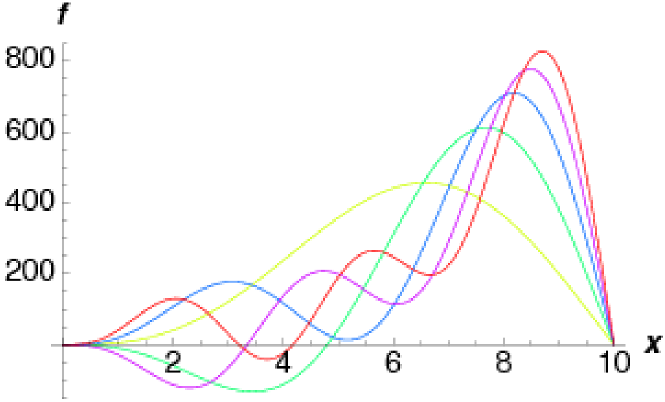


Figure: First 5 partial sums to the Fourier-Bessel series of $f(x)$ using J_3

Chebyshev's differential equation

Consider the following Sturm-Liouville problem on $[-1,1]$:

$$-\frac{d}{dx}\left[\sqrt{1-x^2}\,\frac{d}{dx}y\right]=\lambda\frac{1}{\sqrt{1-x^2}}y, \qquad \left[p(x)=\sqrt{1-x^2}, \quad q(x)=0, \quad w(x)=\tfrac{1}{\sqrt{1-x^2}}\right].$$

The eigenvalues are $\lambda_n=n^2$ for $n=1,2,\dots$, and the eigenfunctions solve Chebyshev's equation:

$$(1-x^2)y''-xy'+n^2y=0.$$

For each n, one solution is a degree-n "Chebyshev polynomial," defined recursively by

$$T_0(x)=1, \qquad T_1(x)=x, \qquad T_{n+1}(x)=2xT_n(x)-T_{n-1}(x).$$

They are orthogonal with repect to the inner product $\langle f,g\rangle=\displaystyle\int_{-1}^{1}\frac{f(x)g(x)}{\sqrt{1-x^2}}\,dx$.

It can be checked that

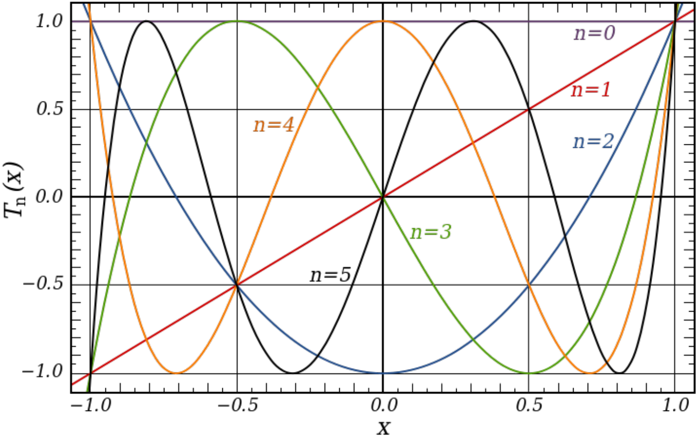
$$\langle T_m,T_n\rangle=\int_{-1}^{1}\frac{T_m(x)T_n(x)}{\sqrt{1-x^2}}\,dx=\begin{cases}\frac{1}{2}\pi\delta_{mn} & m\neq 0, n\neq 0\\ \pi & m=n=0\end{cases}$$

By orthogonality, every function $f(x)$, continuous for $-1<x<1$, can be expressed using Chebyshev polynomials:

$$f(x)\sim\sum_{n=0}^{\infty}c_nT_n(x), \qquad \text{where} \quad c_n=\frac{\langle f,\ T_n\rangle}{\langle T_n,\ T_n\rangle}=\frac{2}{\pi}\langle f,\ T_n\rangle, \ \text{ if } n,m>0.$$

Chebyshev polynomials (of the first kind)

$$T_0(x) = 1 \qquad T_4(x) = 8x^4 - 8x^2 + 1$$
$$T_1(x) = x \qquad T_5(x) = 16x^5 - 20x^3 + 5x$$
$$T_2(x) = 2x^2 - 1 \qquad T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$$
$$T_3(x) = 4x^3 - 3x \qquad T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$$



Hermite's differential equation

Consider the following Sturm-Liouville problem on $(-\infty, \infty)$:

$$-\frac{d}{dx}\left[e^{-x^2}\frac{d}{dx}y\right] = \lambda e^{-x^2}y, \qquad \left[p(x) = e^{-x^2}, \quad q(x) = 0, \quad w(x) = e^{-x^2}\right].$$

The eigenvalues are $\lambda_n = 2n$ for $n = 1, 2, \dots,$ and the eigenfunctions solve Hermite's equation:

$$y'' - 2xy' + 2ny = 0.$$

For each n, one solution is a degree-n "Hermite polynomial," defined by

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2} = \left(2x - \frac{d}{dx}\right)^n \cdot 1$$

They are orthogonal with repect to the inner product $\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)g(x)e^{-x^2}\,dx$.

It can be checked that

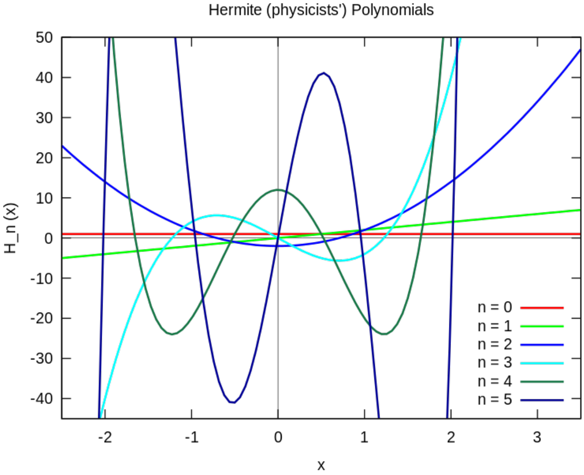
$$\langle H_m, H_n\rangle = \int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2}\,dx = \sqrt{\pi}2^n n!\delta_{mn}.$$

By orthogonality, every function $f(x)$ satisfying $\int_{-\infty}^{\infty} f^2 e^{-x^2}\,dx < \infty$ can be expressed using Hermite polynomials:

$$f(x) \sim \sum_{n=0}^{\infty} c_n H_n(x), \qquad \text{where} \quad c_n = \frac{\langle f,\ H_n\rangle}{\langle H_n,\ H_n\rangle} = \frac{\langle f, H_n\rangle}{\sqrt{\pi}2^n n!}.$$

Hermite polynomials

$$H_0(x) = 1 \qquad H_4(x) = 16x^4 - 48x^2 + 12$$
$$H_1(x) = 2x \qquad H_5(x) = 32x^5 - 160x^3 + 120x$$
$$H_2(x) = 4x^2 - 2 \qquad H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120$$
$$H_3(x) = 8x^3 - 12x \qquad H_7(x) = 128x^7 - 1344x^5 + 3360x^3 - 1680x$$



Hermite functions

The Hermite functions can be defined from the Hermite polynomials as

$$\psi_n(x) = \left(2^n n! \sqrt{\pi}\right)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} H_n(x) = (-1)^n \left(2^n n! \sqrt{\pi}\right)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-x^2}.$$

They are orthonormal with respect to the inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) g(x)\, dx$$

Every real-valued function f such that $\int_{-\infty}^{\infty} f^2\, dx < \infty$ "can be expressed uniquely" as

$$f(x) \sim \sum_{n=0}^{\infty} c_n \psi_n(x)\, dx, \qquad \text{where } c_n = \langle f, \psi_n \rangle = \int_{-\infty}^{\infty} f(x) \psi_n(x)\, dx.$$

These are solutions to the time-independent Schrödinger ODE: $-y'' + x^2 y = (2n+1)y$.