Lecture 6.1: Cauchy problem for the heat and wave equations

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Broad goal

Solve the following initial value problem for heat equation on the real line:

$$u_t = c^2 u_{xx}, \qquad u(x,0) = h(x), \qquad -\infty < x < \infty, \;\; t > 0.$$

This is often called a Cauchy problem.

We will solve this in two steps:

(1) First solve an easier IVP: when initial function is the Heavyside function, $u(x,0) = H(x)$.

(2) Construct a solution to the original IVP using the solution to (1).

The motivation for this approach comes from the following result in dimensional analysis.

Buckingham π theorem (informally)

Every physically meaningful equation involving n physical variables in k dimensional can be rewritten in terms of $p = n - k$ dimensionless parameters $\pi_1, \ldots, \pi_p$.

Example

Consider the simple law from physics: $h = -\dfrac{1}{2}gt^2 + vt$.

Step 1: the Heavyside function

Example 1

Solve the following initial value problem for heat equation on the real line:

$$w_t = c^2 w_{xx}, \qquad w(x,0) = H(x), \qquad -\infty < x < \infty, \;\; t > 0.$$

Example 1 (cont.)

The general solution to the Cauchy problem for heat equation on the real line

$$w_t = c^2 w_{xx}, \qquad w(x,0) = H(x), \qquad -\infty < x < \infty, \quad t > 0.$$

is $w(x,t) = C_1 \int_0^{x/\sqrt{4kt}} e^{-r^2\,dr} + C_2$. Now we'll solve the IVP.

Example 2

Solve the following initial value problem for heat equation on the real line:

$$u_t = c^2 u_{xx}, \qquad w(x,0) = h(x), \qquad -\infty < x < \infty, \;\; t > 0.$$

Remarks

- If w solves the heat equation, so does w_x.
- The function $G(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/(4kt)}$ solves the heat equation.
- The function $G(x-y,t)$ solves the heat equation, and represents an initial unit heat source at y.

Poisson integral representation

Example 2 (summary)

The solution to the initial value problem for heat equation on the real line

$$u_t = c^2 u_{xx}, \qquad w(x,0) = h(x), \qquad -\infty < x < \infty, \;\; t > 0.$$

is $u(x,t) = \displaystyle\int_{-\infty}^{\infty} h(y) \frac{1}{\sqrt{4\pi kt}} e^{-(x-y)^2/(4kt)}\, dy$.

Another way to write this, called the Poisson integral representation, results by substituting $r = \dfrac{x-y}{\sqrt{4kt}}$.

Cauchy problem for the wave equation

Example 3

Solve the following initial value problem for the wave equation on the real line:

$$u_{tt} = c^2 u_{xx}, \qquad u(x,0) = f(x), \qquad u_t(x,0) = g(x), \qquad -\infty < x < \infty, \quad t > 0.$$

Recall that the general solution to $u_{tt} = c^2 u_{xx}$ is

$$u(x,t) = F(x - ct) + G(x + ct),$$

where F and G are arbitrary functions.