Lecture 6.2: Semi-infinite domains and the reflection method

Matthew Macauley

Department of Mathematical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

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Semi-infinite domain, Dirichlet boundary conditions

Example 1

Solve the following B/IVP for the heat equation where $x > 0$ and $t > 0$:

$$u_t = c^2 u_{xx}, \qquad u(0,t) = 0, \qquad u(x,0) = h(x).$$

To solve this, we'll extend $h(x)$ to be an odd function $h_0(x)$:

$$h_0(x) = h(x) \quad \text{if } x > 0, \qquad h_0(x) = -h(-x) \qquad \text{if } x < 0, \qquad h_0(0) = 0.$$

Example 1 (modified)

Solve the following Cauchy problem for the heat equation, where $t > 0$:

$$v_t = c^2 v_{xx}, \qquad v(x,0) = h_0(x).$$

In the previous lecture, we learned that the solution to this Cauchy problem is

$$v(x,t) = \int_{-\infty}^{\infty} h_0(y) G(x-y,t)\, dy, \qquad \text{where} \quad G(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/(4kt)}.$$

Semi-infinite domain, Neumann boundary conditions

Example 2

Solve the following B/IVP for the heat equation where $x > 0$ and $t > 0$: the real line:

$$u_t = c^2 u_{xx}, \qquad u_x(0,t) = 0, \qquad u(x,0) = h(x).$$

To solve this, we'll extend $h(x)$ to be an even function $h_0(x)$:

$$h_0(x) = h(x) \quad \text{if } x \geq 0, \qquad h_0(x) = h(-x) \qquad \text{if } x < 0.$$

Example 2 (modified)

Solve the following Cauchy problem for the heat equation, where $t > 0$:

$$v_t = c^2 v_{xx}, \qquad v(x,0) = h_0(x).$$

As in the previous example, the solution to this Cauchy problem is

$$v(x,t) = \int_{-\infty}^{\infty} h_0(y) G(x-y,t)\, dy, \qquad \text{where} \quad G(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/(4kt)}.$$

The wave equation on a semi-infinite domain

Example 3

Solve the following B/IVP for the wave equation where $x > 0$ and $t > 0$:

$$u_t = c^2 u_{xx}, \qquad u(0,t) = 0, \qquad u(x,0) = f(x), \qquad u_t(x,0) = g(x).$$

Comparing the heat and wave equations on a semi-infinite domain

Dirichlet BCs

The solution to the following B/IVP for the heat equation

$$u_t = c^2 u_{xx}, \qquad u(0,t) = 0, \qquad u(x,0) = h(x)$$

where $x > 0$ and $t > 0$ is

$$u(x,t) = \int_0^\infty \big[G(x-y,t) - G(x+y,t)\big] h(y)dy.$$

The solution to the following B/IVP for the wave equation where

$$u_t = c^2 u_{xx}, \qquad u(0,t) = 0, \qquad u(x,0) = f(x), \qquad u_t(x,0) = g(x).$$

where $x > 0$ and $t > 0$ is

$$u(x,t) = \frac{1}{2}\big(f(x-ct) + f(x+ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds \qquad \text{if } x > ct$$

and

$$u(x,t) = \frac{1}{2}\big(f(x-ct) + f(x+ct)\big) + \frac{1}{2c}\int_{ct-s}^{ct+x} g(s)\,ds \qquad \text{if } 0 < x < ct.$$