

Lecture 6.3: Solving PDEs with Laplace transforms

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Introduction

A function $f: [0, \infty) \rightarrow \mathbb{C}$ has *exponential order*, if $|f(t)| \leq ce^{at}$ holds for sufficiently large t , where $a, c > 0$.

Definition

Let $f: [0, \infty) \rightarrow \mathbb{R}$ be a piecewise continuous function of exponential order. The *Laplace transform* of f is

$$(\mathcal{L}f)(s) = F(s) = \int_0^{\infty} f(t)e^{-st} dt.$$

Property	time-domain	frequency domain
Linearity	$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$
Time / phase-shift	$f(t - c)$	$e^{-cs} F(s)$
Multiplication by exponential	$e^{ct} f(t)$	$F(\omega - c)$
Dilation by $c > 0$	$f(ct)$	$\frac{1}{c} F(s/c)$
Differentiation	$\frac{df(t)}{dt}$	$sF(s) - f(0)$
Multiplication by t	$tf(t)$	$-F'(s)$
Convolution	$f_1(t) * f_2(t)$	$F_1(s) \cdot F_2(s) = (F_1 * F_2)(s)$

Laplace transforms to solve ODEs

Example 1

Solve the initial value problem

$$u'' + u = 0, \quad u(0) = 0, \quad u'(0) = 1.$$

Laplace transform of a multivariate function

Definition

For a function $u(x, t)$ of two variables, define its **Laplace transform** by

$$(\mathcal{L}u)(x, s) = U(x, s) = \int_0^{\infty} u(x, t)e^{st} dt.$$

pause

Remark

The Laplace transform turns t -derivatives into multiplication, and leaves x -derivatives unchanged:

- $(\mathcal{L}u_t)(x, s) = sU(x, s) - u(x, 0)$
- $(\mathcal{L}u_x)(x, s) = U_x(x, s).$

Convolution theorem

For functions f and g ,

$$\mathcal{L}(f * g)(s) = F(s)G(s).$$

By taking the inverse Fourier transform of both sides, it follows that

$$(f * g)(t) = \mathcal{L}^{-1}(F(s)G(s)).$$

Laplace transforms to solve PDEs

Example 2: the diffusion equation on a semi-infinite domain

Let $u(x, t)$ be the concentration of a chemical dissolved in a fluid, where $x > 0$. Consider the following B/IVP problem for the diffusion equation:

$$u_t = u_{xx}, \quad u(0, t) = 1, \quad u(x, 0) = 0.$$

Laplace transforms to solve PDEs

Example 3: the diffusion equation on a semi-infinite domain

Let $u(x, t)$ be the concentration of a chemical dissolved in a fluid, where $x > 0$. Consider the following B/IVP problem for the diffusion equation:

$$u_t = u_{xx}, \quad u(0, t) = f(t), \quad u(x, 0) = 0.$$