Lecture 7.3: The heat and wave equations in higher dimensions

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Overview

Three fundamental PDEs in $\mathbb{R}^n$

- Laplace's equation: $\Delta u = 0$
- Heat equation: $u_t = c^2 \Delta u$
- Wave equation: $u_{tt} = c^2 \Delta u$

Non-Dirichlet and inhomogeneous boundary conditions are more natural for the heat equation.

Solving the heat equation

To solve an B/IVP problem for the heat equation in two dimensions, $u_t = c^2(u_{xx} + u_{yy})$:

1. Find the steady-state solution $u_{ss}(x, y)$ first, i.e., solve Laplace's equation $\Delta u = 0$ with the same BCs.
2. Solve the related heat equation with homogeneous boundary conditions.

Add these two together to get the solution: $u(x, y, t) = u_{ss}(x, y) + u_h(x, y, t)$.

Homogeneous boundary conditions

Example 1a

Solve the following IVP/BVP for the 2D heat equation:

$$u_t = c^2(u_{xx} + u_{yy}), \qquad u(0,y,t) = u(x,0,t) = u(\pi,y,t) = u(x,\pi,t) = 0$$

$$u(x,y,0) = 2\sin x \sin 2y + 3\sin 4x \sin 5y\,.$$

Inhomogeneous boundary conditions

Example 1b

Solve the following IVP/BVP for the 2D heat equation:

$$u_t = c^2(u_{xx} + u_{yy}), \quad u(0,y,t) = u(x,0,t) = u(\pi,y,t) = 0, \quad u(x,\pi,t) = x(\pi - x)$$
$$u(x,y,0) = u_{ss}(x,y) + 2\sin x \sin 2y + 3\sin 4x \sin 5y\,.$$

The wave equation

The set-up

Consider a vibrating square membrane of length L, where the edges are held fixed. If $u(x, y, t)$ is the (vertical) displacement, then u satisfies the following B/IVP for the wave equation:

$$u_{tt} = c^2(u_{xx} + u_{yy}), \qquad u(x,0,t) = u(0,y,t) = u(x,L,t) = u(L,x,t) = 0$$
$$u(x,y,0) = h_1(x,y), \qquad u_t(x,y,0) = h_2(x,y).$$

The functions $h_1(x, y)$ and $h_2(x, y)$ are initial displacement and velocity, respectively.

Finding the general solution

Example 2

Solve the following IVP/BVP for the wave equation:

$$u_{tt} = c^2(u_{xx} + u_{yy}), \qquad u(x,0,t) = u(0,y,t) = u(x,\pi,t) = u(\pi,x,t) = 0$$
$$u(x,y,0) = x(\pi - x)\,y(\pi - y), \qquad u_t(x,y,0) = 0.$$

Solving the resulting IVP

Example 2 (cont.)

The general solution to the following IVP/BVP for the wave equation:

$$u_{tt} = c^2(u_{xx} + u_{yy}), \qquad u(x,0,t) = u(0,y,t) = u(x,\pi,t) = u(\pi,x,t) = 0$$
$$u(x,y,0) = x(\pi - x)\, y(\pi - y), \qquad u_t(x,y,0) = 0.$$

is $u(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \sin mx \, \sin ny \, \cos(c\sqrt{m^2+n^2}\, t)$.