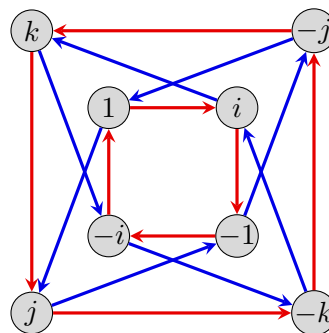
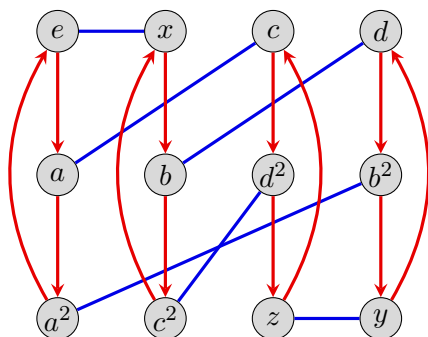


Read the following, which can all be found either in the textbook or on the course website.

- Chapters 5 of *Visual Group Theory* (VGT).
- VGT Exercises 5.3, 5.5, 5.6–5.9, 5.12, 5.14, 5.16, 5.21, 5.25–5.27, 5.29, 5.36, 5.42, 5.44.

Write up solutions to the following exercises.

1. Carry out the following steps for the groups  $A_4$  and  $Q_8$ , whose Cayley graphs are shown below.



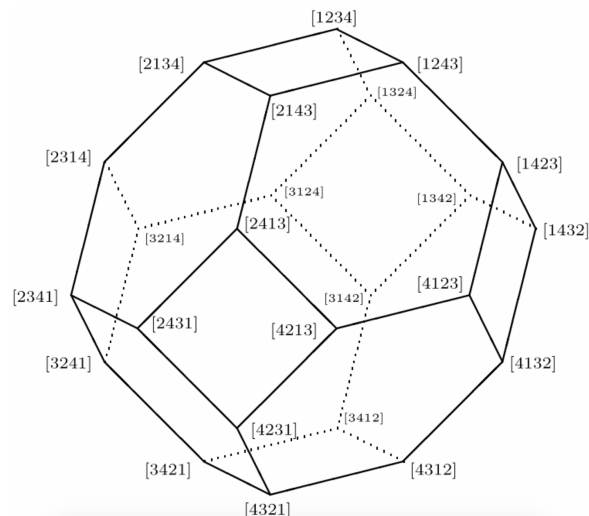
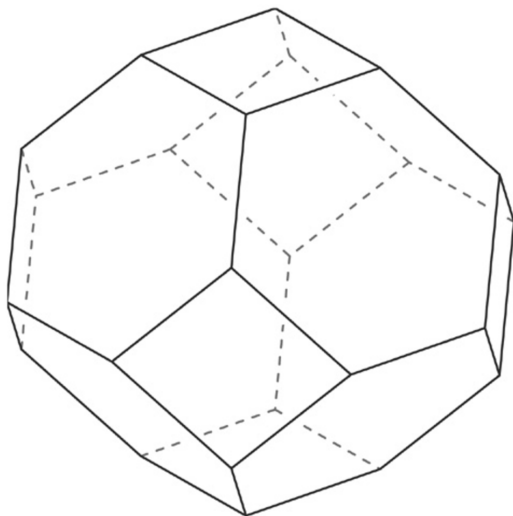
- (a) Find the orbit of each element.
  - (b) Draw the orbit graph of the group.
2. Prove algebraically that if  $g^2 = e$  for every element of a group  $G$ , then  $G$  must be abelian.
  3. Compute the product of the following permutations. Your answer for each should be a single permutation written in cycle notation as a product of disjoint cycles.
    - (a)  $(1\ 3\ 2)(1\ 2\ 5\ 4)(1\ 5\ 3)$  in  $S_5$ ;
    - (b)  $(1\ 5)(1\ 2\ 4\ 6)(1\ 5\ 4\ 2\ 6\ 3)$  in  $S_6$ .
  4. Write out all  $4! = 24$  permutations in  $S_4$  in cycle notation as a product of disjoint cycles. Additionally, write each as a product of transpositions, and decide if they are even or odd. Which of these permutations are also in  $A_4$ ?
  5. (a) The group  $S_3$  can be generated by the transpositions  $(1\ 2)$  and  $(2\ 3)$ . In fact, it has the following presentation

$$S_3 = \langle a, b \mid a^2 = e, b^2 = e, (ab)^3 = e \rangle,$$

where one can take  $a = (1\ 2)$  and  $b = (2\ 3)$ . Make a Cayley diagram for  $S_3$  using this generating set.

- (b) Make a Cayley diagram for the group generated by the permutations  $a = (1\ 2)$  and  $c = (3\ 4)$ , and write down a group presentation for this.

- (c) The group  $S_4$  can be generated by the transpositions  $(1\ 2)$ ,  $(2\ 3)$ , and  $(3\ 4)$ . Make a Cayley diagram for  $S_4$  using this generating set. This can be laid out on a polytope called a *permutahedron*, which is a truncated octahedron shown below. On the right, the vertices are labeled with the corresponding permutations in one-line notation. Make a Cayley graph by labeling the vertices on the unlabeled permutahedron with the 24 permutations of  $S_4$  in cycle notation, and color the edges appropriately.



- (d) Write down a group presentation for  $S_4$  using the generating set in Part (c).
6. The Cayley diagram for  $A_4$  shown above labels the elements with letters instead of permutations:

$$A_4 = \{e, a, a^2, b, b^2, c, c^2, d, d^2, x, y, z\}.$$

Redraw this Cayley diagram but label the nodes with the 12 even permutations from the previous problem. That is, you need to determine which permutation corresponds to  $a$ , which to  $b$ , and so on. [*Hint*: There are many possible ways to do this. If you let  $a$  be one of the permutations of order 3, and let  $x$  be one of the permutations of order 2, then you should be able to determine the remaining elements.]