

Read the following, which can all be found either in the textbook or on the course website.

- Chapters 6 of *Visual Group Theory*, or Chapters 7.3, 8.1 of *IBL Abstract Algebra*.
- VGT Exercises 6.6–6.9, 6.12, 6.17–6.20, 6.28–6.30.

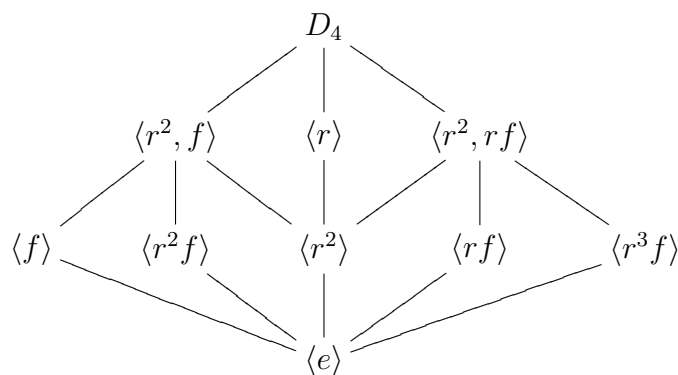
Write up solutions to the following exercises.

1. Consider the subgroups  $H = \langle (123) \rangle$  and  $K = \langle (12)(34) \rangle$  of the alternating group  $A_4 = \langle (123), (12)(34) \rangle$ . Carry out the following steps for both of these subgroups.
  - (a) Write  $A_4$  as a disjoint union of the subgroup's left cosets.
  - (b) Write  $A_4$  as a disjoint union of the subgroup's right cosets.
  - (c) Determine whether the subgroup is normal in  $A_4$ .

2. The *center* of a group  $G$  is the set

$$Z(G) = \{z \in G \mid gz = zg, \forall g \in G\} = \{z \in G \mid gzg^{-1} = z, \forall g \in G\}.$$

- (a) Prove that  $Z(G)$  is a subgroup of  $G$ , and that it is normal in  $G$ .
  - (b) Compute the center of the following groups:  $C_6$ ,  $D_4$ ,  $D_5$ ,  $Q_8$ ,  $A_4$ ,  $S_4$ , and  $D_4 \times Q_8$ .
3. The subgroup lattice of  $D_4$  is shown below:



For each of the 10 subgroups of  $D_4$ , determine whether it is normal in  $D_4$ . Fully justify each yes or no answer.

4. Consider a chain of subgroups  $K \leq H \leq G$ .
  - (a) Prove or disprove: If  $K \triangleleft H \triangleleft G$ , then  $K \triangleleft G$ .
  - (b) Prove or disprove: If  $K \triangleleft G$ , then  $K \triangleleft H$ .
5. Let  $H$  be a subgroup of  $G$ . Given two fixed elements  $a, b \in G$ , define the sets

$$aHbH = \{ah_1bh_2 \mid h_1, h_2 \in H\} \quad \text{and} \quad abH = \{abh \mid h \in H\}.$$

Prove that if  $H \triangleleft G$ , then  $aHbH = abH$ .

6. Prove that  $A \times \{e_B\}$  is a normal subgroup of  $A \times B$ , where  $e_B$  is the identity element of  $B$ . That is, show first that it is a subgroup, and then that it is normal.