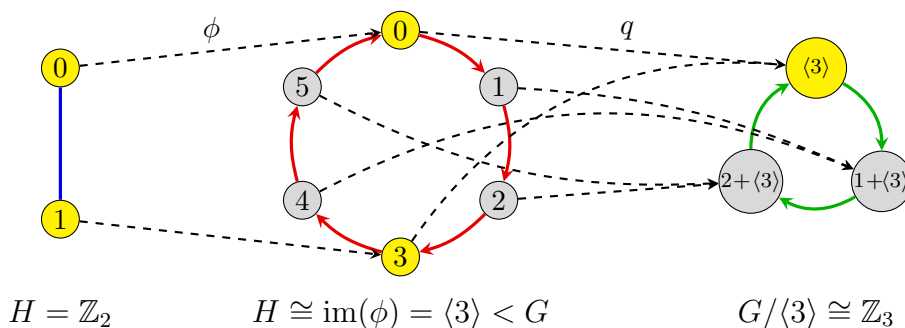


Read the following, which can all be found either in the textbook or on the course website.

- Chapters 8.4–5 of *Visual Group Theory*, or Chapter 9.2 of *IBL Abstract Algebra*.
- VGT Exercises 8.19, 8.23, 8/26, 8.27, 8.37(ab), 8.41, 8.43–8.50.

Write up solutions to the following exercises.

1. Let  $(\mathbb{Q}, +)$  be the group of rational numbers under addition,  $(\mathbb{Q}^*, \cdot)$  the group of non-zero rational numbers under multiplication, and  $(\mathbb{Q}^+, \cdot)$  the group of positive rational numbers under multiplication.
  - (a) Show that  $(\mathbb{Q}^*, \cdot) \cong (\mathbb{Q}^+, \cdot) \times C_2$ . [*Hint*: Recall that  $C_2 = \{e^{0\pi i}, e^{\pi i}\} = \{1, -1\}$ .]
  - (b) Describe the quotient groups  $(\mathbb{Q}, +)/\langle -1 \rangle$  and  $(\mathbb{Q}^*, \cdot)/\langle -1 \rangle$ . In particular, what do the elements (cosets) look like?
  - (c) Use the Fundamental Homomorphism Theorem to prove that  $(\mathbb{Q}^*, \cdot)/\langle -1 \rangle \cong (\mathbb{Q}^+, \cdot)$ .
  
2. For Parts (a)–(d), a group  $G$  is given together with a normal subgroup  $H$ . Illustrate the embedding  $\phi: H \rightarrow G$ , and the quotient map  $q: G \rightarrow G/H$ , chained together so that  $\text{im}(\phi) = \ker(q)$ . An example for  $G = \mathbb{Z}_6$  and  $H = \mathbb{Z}_2$  is shown below:



- (a)  $G = \mathbb{Z}_6$ ,  $H = \mathbb{Z}_3$ ,
- (b)  $G = D_3$ ,  $H = C_3$ ,
- (c)  $G = A_4$ ,  $H = V_4$ ,
- (d)  $G = S_n$ ,  $H = A_n$  [don't draw the actual Cayley graphs for this one, just the maps].

Now, answer each of the following questions about each of your answers to Parts (a)–(d).

- (e) What map  $\theta$  into  $H$  would satisfy the equation  $\text{im } \theta = \ker \phi$ ? Choose one with the smallest possible domain.
- (f) What map  $\theta'$  from  $G/H$  would satisfy the equation  $\text{im}(q) = \ker(\theta')$ ? Choose one with the smallest possible codomain.
- (g) Add the two maps  $\theta$  and  $\theta'$  to your illustration.
- (h) The new chain of four homomorphisms is called a *short exact sequence*. It is one way to use homomorphisms to illustrate quotients, and it shows a connection between embeddings and quotient maps. Given a normal subgroup  $H \triangleleft G$ , show how to create a short exact sequence involving  $G$  and  $H$ .

3. Let  $A$  and  $B$  be normal subgroups of  $G$ . In this problem, you will prove the *Diamond Isomorphism Theorem*.

- Prove that the set  $AB := \{ab : a \in A, b \in B\}$  is a subgroup of  $G$ .
- Prove that  $B \triangleleft AB$  and  $A \cap B \triangleleft A$ .
- Prove that  $A/(A \cap B) \cong AB/B$ . [*Hint*: Construct a homomorphism  $\phi: A \rightarrow AB/B$  that has kernel  $A \cap B$ , then apply the FHT.]
- Draw a diagram, or lattice, of  $G$  and its subgroups  $AB$ ,  $A$ ,  $B$ , and  $A \cap B$ . Interpret the result in Part (c) in terms of this diagram.

4. For each part below, consider the group  $G = \langle A, B \rangle$  generated by the two matrices given. Assume that matrix multiplication is the binary operation, and  $i = \sqrt{-1}$ . To what common group is  $G$  isomorphic? Write down an explicit isomorphism (you only need to define it for the generators), and a group presentation for  $G$ .

$$(a) \quad A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

$$(b) \quad A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

$$(c) \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

5. In this exercise, you will prove that if  $A$  and  $B$  are normal subgroups and  $AB = G$ , then

$$G/(A \cap B) \cong (G/A) \times (G/B).$$

(a) Consider the following map:

$$\phi: AB \longrightarrow (G/A) \times (G/B), \quad \phi(g) = (gA, gB).$$

Show that  $\phi$  is a homomorphism.

- Show that  $\phi$  is surjective. That is, given any  $(g_1A, g_2B)$ , show that there is some  $g = ab \in AB$  such that  $\phi(g) = (g_1A, g_2B)$ . [*Hint*: Try  $g = a_2b_1$ .]
- Find  $\ker(\phi)$  [you need to prove your answer is correct] and then apply the Fundamental Homomorphism Theorem.

6. For each order given below, list all abelian groups of that order by writing each one as a product of cyclic groups of prime power order. Additionally, write each one as a product of cyclic groups organized by “elementary divisors.”

- 8
- 54
- 400
- $p^2q$ , where  $p$  and  $q$  are distinct primes.