

Read the following, which can all be found either in the textbook or on the course website.

- Chapter 9.1 of *Visual Group Theory* (VGT).
- VGT Exercises 8.15–8.18, 9.17.

Write up solutions to the following exercises.

1. The commutator subgroup of a group G is the subgroup

$$G' = \langle aba^{-1}b^{-1} \mid a, b \in G \rangle.$$

- (a) Prove that G is abelian if and only if $G' = \{e\}$.
- (b) Prove that $G' \triangleleft G$. [*Hint*: Take a “commutator” $c = aba^{-1}b^{-1}$ and prove that $gcg^{-1} \in G'$.]
- (c) Prove that G' is the intersection of all normal subgroups of G that contain the set $C := \{aba^{-1}b^{-1} \mid a, b \in G\}$:

$$G' = \bigcap_{C \subseteq N \triangleleft G} N$$

- (d) If we quotient G by G' , then we are in essence, “killing” all non-abelian parts of the Cayley diagram, as shown below:



Prove algebraically that G/G' is indeed abelian.

2. For each of the following groups G , compute its commutator subgroup G' and its abelianization G/G' . Finally, draw the subgroup lattice of G and circle every normal subgroup, and circle twice the one that is G' .
 - (a) V_4 ,
 - (b) D_3 ,
 - (c) Q_8 .
3. Find the commutator subgroup of each of the following groups and compute its abelianization.
 - (a) An abelian group A .
 - (b) The alternating group A_n , for $n \geq 5$. [*Hint*: A_n is a *simple group*, which means its only normal subgroups are $\langle e \rangle$ and A_n .]
 - (c) The dihedral group D_n for n even.
 - (d) The dihedral group D_n for n odd.

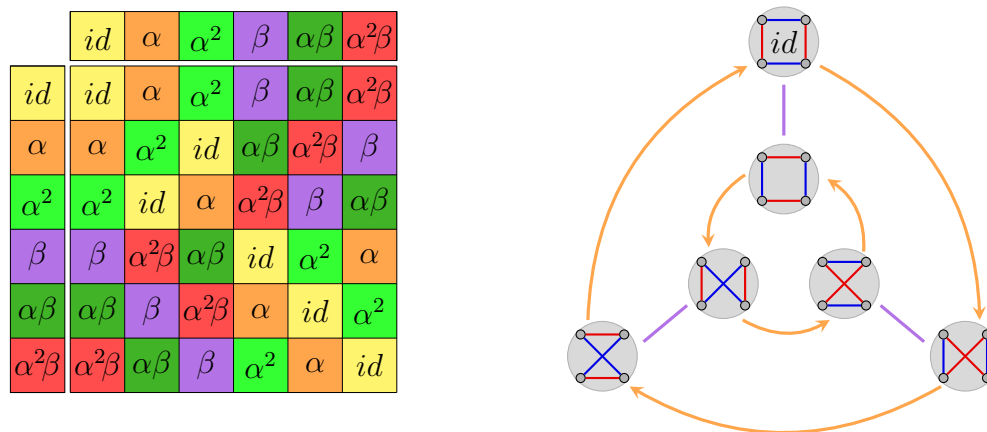
4. Recall that the automorphism group of $V_4 = \langle h, v \rangle = \{e, h, v, r\}$, where $r = hv$, has presentation

$$\text{Aut}(V_4) = \langle \alpha, \beta \mid \alpha^3 = \beta^2 = (\alpha\beta)^2 = id \rangle, \quad \text{where} \quad \begin{array}{l} h \xrightarrow{\alpha} v \\ v \mapsto r \end{array} \quad \text{and} \quad \begin{array}{l} h \xrightarrow{\beta} v \\ v \mapsto h. \end{array}$$

The generating automorphisms are the following permutations of V_4 :

$$\alpha : \quad e \quad h \xrightarrow{\alpha} v \xrightarrow{\alpha} r \quad \text{and} \quad \beta : \quad e \quad h \xrightarrow{\beta} v \quad r$$

The multiplication table and Cayley diagram of $\text{Aut}(V_4) = \langle \alpha, \beta \rangle$, which highlights how automorphisms are “re-wirings”, are shown below:



Repeat the above steps for each of the following groups. Use the Cayley diagram defined by the generating set given. Recall that $\text{Aut}(\mathbb{Z}_n) \cong U_n$.

- (a) $\mathbb{Z}_5 = \langle 1 \rangle$,
 - (b) $\mathbb{Z}_6 = \langle 1 \rangle$,
 - (c) $\mathbb{Z}_2 \times \mathbb{Z}_3 = \langle (1, 0), (0, 1) \rangle$, [Recall that $\mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$]
 - (d) $\mathbb{Z}_8 = \langle 1 \rangle$.
5. Let G act on a set S . Prove that $\text{Stab}(s)$ is a subgroup of G for every $s \in S$.
6. Suppose the cyclic group C_5 acts on a set $S = \{A, B, C, D\}$.
- (a) What are the possible sizes of the orbits?
 - (b) What are the possible stabilizer subgroups of each orbit?
 - (c) Draw the action diagram.

Fully explain your reasoning for each part.