

- For each of the following rings  $R$ , determine the zero divisors (right and left, if appropriate), and the set  $U(R)$  of units.
  - The set  $\mathcal{C}^1$  of continuous real-valued functions  $f: \mathbb{R} \rightarrow \mathbb{R}$ .
  - The polynomial ring  $\mathbb{R}[x]$ .
  - $\mathbb{Z} \times \mathbb{Z}$ , where addition and multiplication are defined componentwise.
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- Prove that if a left ideal  $I$  of a ring  $R$  contains a unit, then  $I = R$ .
- Let  $I$  and  $J$  be ideals of a ring  $R$ .
  - Prove that  $I + J$ ,  $I \cap J$ , and  $IJ$  are ideals of  $R$ .
  - If  $R$  is commutative, then the set  $(I : J) = \{r \in R \mid rJ \subseteq I\}$  is called the *ideal quotient* or *colon ideal* of  $I$  and  $J$ . Show that  $(I : J)$  is an ideal of  $R$ .
  - Consider the ideals  $I = 4\mathbb{Z}$  and  $J = 6\mathbb{Z}$  of the ring  $R = \mathbb{Z}$ . Compute  $I + J$ ,  $I \cap J$ ,  $IJ$ ,  $(I : J)$ , and  $(J : I)$ .
  - Repeat Part (c) for the ideals  $I = m\mathbb{Z}$  and  $J = n\mathbb{Z}$  of  $R = \mathbb{Z}$ .
- The left ideal generated by  $X \subseteq R$  is defined as

$$(X) := \bigcap \{I \mid I \text{ is a left ideal s.t. } X \subseteq I \subseteq R\}.$$

- Prove that the left ideal generated by  $X$  is

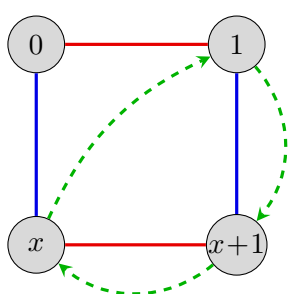
$$(X) = \{r_1x_1 + \cdots + r_nx_n \mid n \in \mathbb{N}, r_i \in R, x_i \in X\}.$$

- The two-sided ideal generated by  $X \subseteq R$  is defined by replacing “left” with “two-sided” in the definition above. Prove this is also equal to

$$\{r_1x_1s_1 + \cdots + r_nx_ns_n \mid n \in \mathbb{N}, r_i, s_i \in R, x_i \in X\}.$$

- Find a (non-commutative) ring  $R$  and a set  $X$  such that the left and two-sided ideals generated by  $X$  are different.

- The finite field  $\mathbb{F}_4$  on 4 elements can be constructed as the quotient of the polynomial  $\mathbb{Z}_2[x]$  by the ideal  $I = (x^2 + x + 1)$  generated by the irreducible polynomial  $x^2 + x + 1$ . The figure below shows a Cayley diagram, and multiplication and addition tables for the finite field  $\mathbb{Z}_2[x]/(x^2 + x + 1) \cong \mathbb{F}_4$ .



+	0	1	$x$	$x+1$
0	0	1	$x$	$x+1$
1	1	0	$x+1$	$x$
$x$	$x$	$x+1$	0	1
$x+1$	$x+1$	$x$	1	0

×	1	$x$	$x+1$
1	1	$x$	$x+1$
$x$	$x$	$x+1$	1
$x+1$	$x+1$	1	$x$

- (a) Find a degree-3 polynomial  $f \in \mathbb{Z}_2[x]$  that is irreducible over  $\mathbb{Z}_2$ , and a degree-2 polynomial  $g \in \mathbb{Z}_3[x]$  that is irreducible over  $\mathbb{Z}_3$ . [*Hint*: Any polynomial with no roots in the “prime field”  $\mathbb{Z}_p$  will work.]
- (b) Construct Cayley diagrams, addition, and multiplication tables for the finite fields

$$\mathbb{F}_8 \cong \mathbb{Z}_2[x]/(f) \quad \text{and} \quad \mathbb{F}_9 \cong \mathbb{Z}_3[x]/(g).$$

6. Prove the Fundamental Homomorphism Theorem (FHT) for rings: If  $\phi: R \rightarrow S$  is a ring homomorphism, then  $\text{Ker } \phi$  is a two-sided ideal of  $R$ , and  $R/\text{Ker } \phi \cong \text{Im } \phi$ . You may assume the FHT for groups.