- 1. Let $p \in \mathbb{N}$ be a fixed prime. For each of the three ideals I = (p), (x), and (x, p) in the ring $R = \mathbb{Z}[x]$, do the following steps:
 - (i) Describe the elements of the ideal formally, as $I = \{$: $\}$.
 - (ii) Characterize the polynomials in I in plain English.
 - (iii) Determine whether I is maximal and/or prime.
 - (iv) Describe the quotient ring R/I.

Then, repeat the above steps for these ideals but in the ring $\mathbb{Q}[x]$.

- 2. Let R be a commutative ring with 1.
 - (a) Prove that R is an integral domain if and only if 0 is a prime ideal.
 - (b) Prove that an ideal $P \subseteq R$ is prime if and only if R/P is an integral domain.
 - (c) Show that every maximal ideal is prime.
- 3. Let p be a fixed prime number, and consider the ring

$$R = \left\{ \frac{a}{b} : a, b \in \mathbb{Z}, \ \gcd(a, b) = 1, \ p \nmid b \right\}.$$

Find the group of units U(R), and all maximal ideals of R. Justify your answers.

- 4. Recall that $a, b \in R$ are associates, denoted $a \sim b$, if $a \mid b$ and $b \mid a$. Show that $a \sim b$ if and only if a = bu for some unit $u \in R$.
- 5. Let R be a principal ideal domain (PID). A common multiple of $a, b \in R^*$ is an element m such that a|m and b|m. Moreover, m is a least common multiple (LCM) if m|n for any other common multiple n of a and b.
 - (a) Prove that any $a, b \in R^*$ have an LCM.
 - (b) Prove that an LCM of a and b is unique up to multiplication of associates, and can be characterized as a generator of the (principal) ideal $I := (a) \cap (b)$.
- 6. For any $x = r + s\sqrt{m} \in \mathbb{Q}(\sqrt{m})$, define the *norm* of x to be $N(x) = r^2 ms^2$.
 - (a) Show that N(xy) = N(x)N(y).
 - (b) Show that $N(x) \in \mathbb{Z}$ if $x \in R_m$.
 - (c) Show that $u \in U(R_m)$ if and only if |N(u)| = 1.
 - (d) Show that $U(R_{-1}) = \{\pm 1, \pm i\}$, $U(R_{-3}) = \{\pm 1, \pm (1 \pm \sqrt{3})/2\}$, and $U(R_m) = \{\pm 1\}$ for all other negative square-free $m \in \mathbb{Z}$.
- 7. Let $R = \mathbb{Z}_{10}$ and $D = \{0, 2, 4, 6, 8\} \subset R$. Find the field of fractions of D in R.