Lecture 2.2: Dihedral groups

Matthew Macauley

Department of Mathematical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

Math 4120, Modern Algebra

Overview

In this series of lectures, we are introducing 5 families of groups:

1. cyclic groups
2. abelian groups
3. dihedral groups
4. symmetric groups
5. alternating groups

This lecture is focused on the third family: dihedral groups.

These are the groups that describe the symmetry of regular n-gons.

Dihedral groups

While cyclic groups describe 2D objects that only have rotational symmetry, dihedral groups describe 2D objects that have rotational *and* reflective symmetry.

Regular polygons have rotational and reflective symmetry. The dihedral group that describes the symmetries of a regular n-gon is written D_n.

All actions in C_n are also actions of D_n, but there are more than that. The group D_n contains $2n$ actions:

- n rotations
- n reflections.

However, we only need two generators. Here is one possible choice:

1. $r =$ counterclockwise rotation by $2\pi/n$ radians. (A single "click.")
2. $f =$ flip (fix an axis of symmetry).

Here is one of (of many) ways to write the $2n$ actions of D_n:

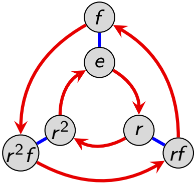
$$D_n = \{\underbrace{e, r, r^2, \dots, r^{n-1}}_{\text{rotations}}, \underbrace{f, rf, r^2f, \dots, r^{n-1}f}_{\text{reflections}}\}.$$

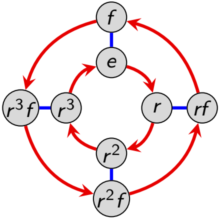
Cayley diagrams of dihedral groups

Here is one possible presentation of D_n:

$$D_n = \langle r, f \mid r^n = e,\ f^2 = e,\ rfr = f \rangle.$$

Using this generating set, the Cayley diagrams for the dihedral groups all look similar. Here they are for D_3 and D_4, respectively.

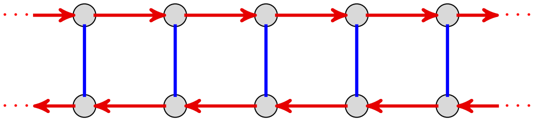




There is a related infinite dihedral group D_∞, with presentation

$$D_\infty = \langle r, f \mid f^2 = e,\ rfr = f \rangle.$$

We have already seen its Cayley diagram:



Cayley diagrams of dihedral groups

If s and t are two reflections of an n-gon across adjacent axes of symmetry (i.e., axes incident at π/n radians), then st is a rotation by $2\pi/n$.

To see an explicit example, take $s = rf$ and $t = f$ in D_n; obviously $st = (rf)f = r$.

Thus, D_n can be generated by two reflections. This has group presentation

$$\begin{aligned} D_n &= \langle s, t \mid s^2 = e,\ t^2 = e,\ (st)^n = e \rangle \\ &= \{\underbrace{e, st, ts, (st)^2, (ts)^2, \ldots,}_{\text{rotations}}\ \underbrace{s, t, sts, tst, \ldots}_{\text{reflections}}\}. \end{aligned}$$

What would the Cayley diagram corresponding to this generating set look like?

Remark

If $n \geq 3$, then D_n is nonabelian, because $rf \neq fr$. However, the following relations are very useful:

$$rf = fr^{n-1}, \qquad fr = r^{n-1}f.$$

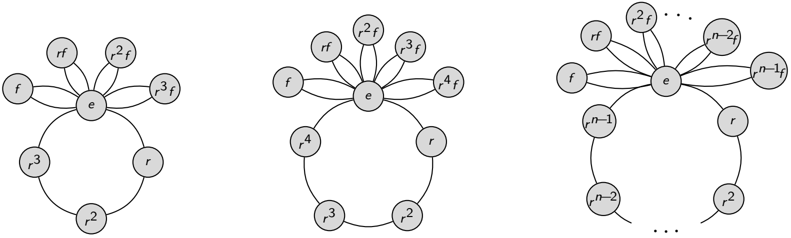
Looking at the Cayley graph should make these relations visually obvious.

Cycle graphs of dihedral groups

The (maximal) orbits of D_n consist of

- 1 orbit of size n consisting of $\{e, r, \ldots, r^{n-1}\}$;
- n orbits of size 2 consisting of $\{e, r^k f\}$ for $k = 0, 1, \ldots, n-1$.

Here is the general pattern of the cycle graphs of the dihedral groups:



Note that the size-n orbit may have smaller subsets that are orbits. For example, $\{e, r^2, r^4, \ldots, r^{n-2}\}$ and $\{e, r^{n/2}\}$ are orbits if n is even.

Multiplication tables of dihedral groups

The separation of D_n into rotations and reflections is also visible in their multiplication tables. For example, here is D_4:

	e	r	r^2	r^3	f	rf	r^2f	r^3f
e	e	r	r^2	r^3	f	rf	r^2f	r^3f
r	r	r^2	r^3	e	rf	r^2f	r^3f	f
r^2	r^2	r^3	e	r	r^2f	r^3f	f	rf
r^3	r^3	e	r	r^2	r^3f	f	rf	r^2f
f	f	r^3f	r^2f	rf	e	r^3	r^2	r
rf	rf	f	r^3f	r^2f	r	e	r^3	r^2
r^2f	r^2f	rf	f	r^3f	r^2	r	e	r^3
r^3f	r^3f	r^2f	rf	f	r^3	r^2	r	e

	e	r	r^2	r^3	f	rf	r^2f	r^3f
e	e	r	r^2	r^3	f	rf	r^2f	r^3f
r	r	r^2	r^3	e	rf	r^2f	r^3f	f
r^2	r^2	r^3	e	r	r^2f	r^3f	f	rf
r^3	r^3	e	r	r^2	r^3f	f	rf	r^2f
f	f	r^3f	r^2f	rf	e	r^3	r^2	r
rf	rf	f	r^3f	r^2f	r	e	r^3	r^2
r^2f	r^2f	rf	f	r^3f	r^2	r	e	r^3
r^3f	r^3f	r^2f	rf	f	r^3	r^2	r	e

non-flip **flip**

flip **non-flip**

As we shall see later, the partition of D_n as depicted above forms the structure of the group C_2. "Shrinking" a group in this way is called taking a quotient.

It yields a group of order 2 with the following Cayley diagram:

	e	f
e	e	f
f	f	e