

Lecture 2.3: Symmetric and alternating groups

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Overview

In this series of lectures, we are introducing 5 families of groups:

1. cyclic groups
2. abelian groups
3. dihedral groups
4. symmetric groups
5. alternating groups

This lecture is focused on the last two families: **symmetric groups** and **alternating groups**.

Loosely speaking, a symmetric group is the collection of all $n!$ permutations of n objects. Alternating groups are similar.

Thus, we will study permutations, and how to write them concisely in **cycle notation**.

This is important because Cayley's theorem tells us that every finite group is isomorphic to a collection of permutations (i.e., a **subgroup** of a symmetric group.)

Symmetric groups

Most groups we have seen have been collections of ways to rearrange things. This can be formalized.

Definition

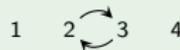
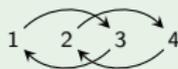
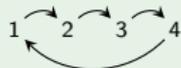
A **permutation** is an action that rearranges a collection of things.

For convenience, we will usually refer to permutations of positive integers (just like we did when we numbered our rectangle, etc.).

There are many ways to represent permutations, but we will start with the notation illustrated by the following example.

Example

Here are some permutations of 4 objects.



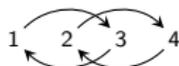
Combining permutations

In order for the set of permutations of n objects to form a group (what we want!), we need to understand how to combine permutations. Let's consider an example.

What should

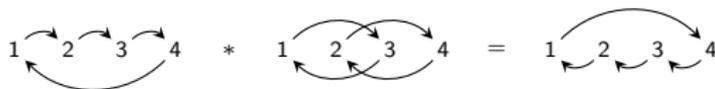


followed by



be equal to?

The first permutation rearranges the 4 objects, and then we shuffle the result according to the second permutation:



Groups of permutations

Fact

There are $n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$ permutations of n items.

For example, there are $4! = 24$ “permutation pictures” on 4 objects.

The collection of permutations of n items forms a group!

To verify this, we just have to check that the appropriate rules of *one* of our definitions of a group hold.

How do we find the inverse of a permutation? Just reverse all of the arrows in the permutation picture. For example, the inverse of



is simply



The symmetric group

Definition

The group of all permutations of n items is called the **symmetric group** (on n objects) and is denoted by S_n .

We've already seen the group S_3 , which happens to be the same as the dihedral group D_3 , but this is the only time the symmetric groups and dihedral groups coincide. (*Why?*)

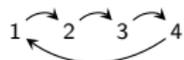
Although the set of *all* permutations of n items forms a group, creating a group does not require taking all permutations.

If we choose carefully, we can form groups by taking a subset of the permutations.

For example, the cyclic group C_n and the dihedral group D_n can both be thought of groups of certain permutations of $\{1, \dots, n\}$. (*Why? Do you see which permutations they represent?*)

Cycle notation for S_n

We can concisely describe the permutation



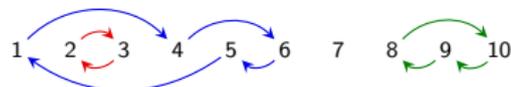
as $(1\ 2\ 3\ 4)$.

This is called **cycle notation**.

Observation 1

Every permutation can be decomposed into a product of **disjoint cycles**.

For example, in S_{10} , we can write



as $(1\ 4\ 6\ 5)(2\ 3)(8\ 10\ 9)$.

Observation 2

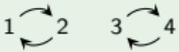
Disjoint cycles commute.

For example: $(1\ 4\ 6\ 5)(2\ 3)(8\ 10\ 9) = (2\ 3)(8\ 10\ 9)(1\ 4\ 6\ 5)$.

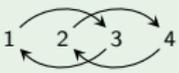
Cycle notation for S_n

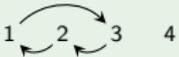
Example

Consider the following permutations in S_4 :

■  is $(1\ 2)(3\ 4)$

■  is $(2\ 3)$

■  is $(1\ 3)(2\ 4)$

■  is $(1\ 3\ 2)$

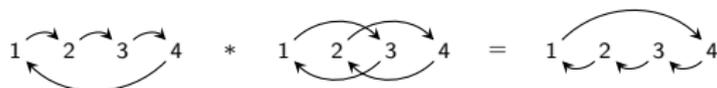
Remark

It doesn't matter "where we start" when writing the cycle. In the last example above,

$$(1\ 3\ 2) = (3\ 2\ 1) = (2\ 1\ 3) = (1\ 2)(2\ 3) = (1\ 2)(2\ 3)(2\ 3)(2\ 3).$$

Composing permutations in cycle notation

Recall how we combined permutations:



In cycle notation, this is

$$(1\ 2\ 3\ 4) * (1\ 3)(2\ 4) = (1\ 4\ 3\ 2).$$

We read **left-to-right**. (*Caveat*: some books use the right-to-left convention as in function composition.)

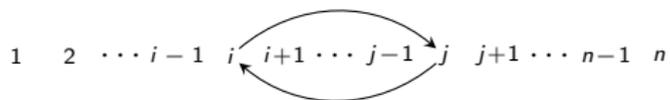
Do you see how to combine permutations in cycle notation? In the example above, we start with 1 and then read off:

- “1 goes to 2, then 2 goes to 4”; Write: (1 4
- “4 goes to 1, then 1 goes to 3”; Write: (1 4 3
- “3 goes to 4, then 4 goes to 2”; Write: (1 4 3 2
- “2 goes to 3, then 3 goes to 1”; Write: (1 4 3 2)

In this case, we’ve used up each number in $\{1, \dots, n\}$. If we hadn’t, we’d take the smallest unused number and continue the process with a new (disjoint) cycle.

Transpositions

A **transposition** is a permutation that swaps two objects and fixes the rest, e.g.:



In cycle notation, a transposition is just a 2-cycle, e.g., $(i \ j)$.

Theorem

The group S_n is generated by transpositions.

Intuitively, this means that every permutation can be constructed by successively exchanging pairs of objects.

In other words, if n people are standing in a row, and we want to rearrange them in some other order, we can always do this by successively having pairs of people swap places.

In fact, we only need **adjacent transpositions** to generate S_n :

$$S_n = \langle (1 \ 2), (2 \ 3), \dots, (n-1 \ n) \rangle.$$

Transpositions and the alternating groups

Remark

Even though every permutation in S_n can be written as a product of transpositions, there may be many ways to do this.

For example:

$$(1\ 3\ 2) = (1\ 2)(2\ 3) = (1\ 2)(2\ 3)(2\ 3)(2\ 3) = (1\ 2)(2\ 3)(1\ 2)(1\ 2).$$

Theorem

The **parity** of the number of transpositions of a fixed permutation is unique.

That is, a fixed permutation can either be written with an **even number** of transpositions, or an **odd number** of transpositions, but *not both!*

We thus have a notion of **even permutations** and **odd permutations**.

Theorem

Exactly half of the permutations in S_n are even, and they form a group called the **alternating group**, denoted A_n .

Practice

At this point, it helps to “get your hands dirty” and try a few examples. Here are some good exercises.

1. Write the following products of permutations into a product of *disjoint cycles*:
 - $(1\ 2\ 3)(1\ 2\ 3\ 4)$ in S_4
 - $(1\ 6)(1\ 2\ 4\ 5)(1\ 6\ 4\ 2\ 5\ 3)$ in S_6 .

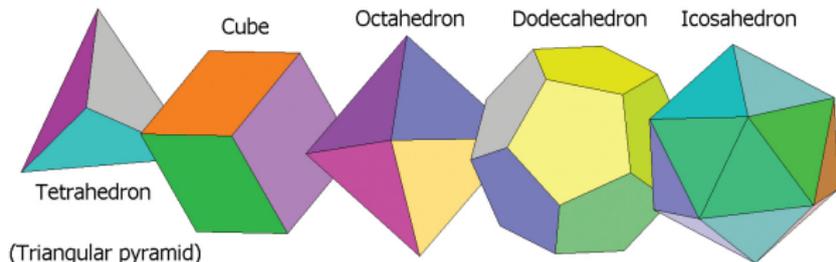
Let $G = S_3$, the symmetric group on three objects. This group has six elements.

2. Do the following for each element in S_3 :
 - Draw its “permutation picture.”
 - Write it as a product of disjoint transpositions (that is, using only $(1\ 2)$, $(2\ 3)$, and $(1\ 3)$).
 - Write it as a product of disjoint *adjacent* transpositions (that is, using only $(1\ 2)$ and $(2\ 3)$).
 - Determine whether it is even or odd.
3. Now, write down the alternating group A_3 . This is the group consisting of only the *even* permutations. What familiar group is this isomorphic to?

Platonic solids

The symmetric groups and alternating groups arise throughout group theory. In particular, the groups of symmetries of the 5 **Platonic solids** are symmetric and alternating groups.

There are only *five* 3-dimensional shapes (polytopes) all of whose faces are regular polygons that meet at equal angles. These are called the Platonic solids:



The groups of symmetries of the Platonic solids are as follows:

shape	group
Tetrahedron	A_4
Cube	S_4
Octahedron	S_4
Icosahedron	A_5
Dodecahedron	A_5

Platonic solids

The Cayley diagrams for these 3 groups can be arranged in some very interesting configurations.

In particular, the Cayley diagram for Platonic solid 'X' can be arranged on a truncated 'X', where truncated refers to cutting off some corners.

For example, here are two representations for Cayley diagrams of A_5 . At left is a truncated **icosahedron** and at right is a truncated **dodecahedron**.

