Lecture 1.2: Linear independence and spanning sets

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Linear independence

Definition (recall)

A vector space consists of a set V (of "vectors") and a set $\mathbb{F}$ (of "scalars"; usually $\mathbb{R}$ or $\mathbb{C}$) that is:

- closed under addition: $v, w \in V \implies v + w \in V$
- closed under scalar multiplication: $v \in V$, $c \in \mathbb{F} \implies cv \in V$

In general, we are <u>not</u> allowed to multiply vectors.

Definition

A set $S \subseteq V$ is linearly independent if for any $v_1, \ldots, v_n \in S$:

$$a_1 v_1 + \cdots + a_n v_n = 0 \quad \implies \quad a_1 = a_2 = \cdots = a_n = 0.$$

If S is not linearly independent, then it is linearly dependent.

Intuition

$S \subseteq V$ is linearly independent if none of the vectors in S can be expressed as a linear combination of the others.

Linear independence

Definition (recall)

A set $S \subseteq V$ is linearly independent if for any $v_1, \dots, v_n \in S$:

$$a_1 v_1 + \cdots + a_n v_n = 0 \quad \Longrightarrow \quad a_1 = a_2 = \cdots = a_n = 0.$$

Example 1

Let $V = \mathbb{R}^3$, and $S \subseteq V$.

1. The set $S = \{v_1\}$ is linearly independent iff $v_1 \neq 0$.
2. The set $S = \{v_1, v_2\}$ is linearly independent iff v_1 and v_2 don't lie on the same line.
3. The set $S = \{v_1, v_2, v_3\}$ is linearly independent iff v_1, v_2, v_3 don't lie on the same plane.
4. The set $S = \{v_1, v_2, v_3, v_4\}$ is *never* linearly independent in $\mathbb{R}^3$.

Example 2

Let $V = \mathbb{R}_3[x]$, and $S \subseteq V$.

1. The set $S = \{1, x, x^2\}$ is linearly independent.
2. The set $S = \{1, x, x^2, x^3\}$ is linearly independent.
3. The set $S = \{1, x, x^2, 1 + 3x - 4x^2\}$ is linearly dependent.
4. The set $S = \{1, x, x^2, x^3 + x + 1\}$ is linearly independent.

Linear independence

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Example 3

Let $V = \mathcal{C}^\infty(\mathbb{C})$, and $S \subseteq V$.

1. $S = \{\cos t, \sin t\}$ is linearly independent.

   Reason: If $C_1 \cos t + C_2 \sin t = 0$, then $C_1 = C_2 = 0$.

2. $S = \{e^{2t}, e^{3t}\}$ is linearly independent.

   Reason: If $C_1 e^{2t} + C_2 e^{3t} = 0$, then $C_1 = C_2 = 0$.

3. $S = \{e^{2it}, e^{-2it}, \cos 2t\}$ is linearly dependent.

   Reason: $\cos 2t = \frac{1}{2} e^{2it} + \frac{1}{2} e^{-2it}$.

4. $S = \{e^{2t}, e^{-2t}, \cosh 2t\}$ is linearly dependent.

   Reason: $\cosh 2t = \frac{1}{2} e^{2t} + \frac{1}{2} e^{-2t}$.

Spanning sets and bases

Definition

A subset $S \subseteq V$ spans V if every $v \in V$ can be written as $v = a_1 v_1 + \cdots + a_n v_n$ where $v_i \in S$, $a_i \in \mathbb{F}$.

Moreover, if S is also linearly independent then S is a basis of V.

Intuition

- "*S spans V*" means "*S generates all of V*"
- "*S is a basis for V*" means "S is a minimal set that generates V."

Examples

Let $V = \mathbb{R}^2$.

	Spans $\mathbb{R}^2$?	Basis for $\mathbb{R}^2$?
$S = \{(1,0),\, (0,1)\}$		
$S = \{(3,1),\, (1,1)\}$		
$S = \{(1,0),\, (0,1),\, (3,1)\}$		
$S = \{(1,1)\}$		

Spanning sets and bases

Theorem

Let $S \subseteq V$. The following are equivalent:

- S is a basis of V,
- S is a minimal spanning set of V,
- S is a maximal linearly independent set in V.

Example. Let $V = \mathbb{R}^3$, $W \subseteq V$ any plane (through $\mathbf{0}$).

Intuition. We need two vectors (not collinear) to generate W.

In fact, $S = \{v_1, v_2\}$ is a *basis* for W iff v_1 and v_2 are not collinear.

Let's "go up" a dimension and find a basis for V.

$S = \{v_1, v_2, v_3\}$ is a basis for V iff they are not co-planar.

It should be clear how this generalizes to higher dimensions.

(By the way, what do we mean by "*dimension*"?)

Spanning sets and bases

Definition

The dimension of a vector space is the number of vectors in any basis.

Examples

- $\dim(\mathbb{R}^n) = n$: Basis: $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$
- $\dim(\mathbb{R}_n[x]) = n + 1$: Basis: $\{1, x, \ldots, x^n\}$
- $\dim(\mathbb{R}[x]) = \infty$: Basis: $\{1, x, x^2, \ldots\}$
- $\dim(\mathrm{Per}_{2\pi}) = \infty$: Basis: $\{1, \cos x, \cos 2x, \ldots\} \cup \{\sin x, \sin 2x, \ldots\}$.

Remark

Any subset $S \subseteq V$ spans a subspace W of V. Denote this subspace by $\mathrm{Span}(S)$.

How to construct a basis from a spanning set

Algorithm

Consider any finite subset $S \subseteq V$ and let $W = \text{Span}(S)$.

We may ask: *If S a basis for W?*

If not, then S is not a minimal spanning set, so we can remove some v_1 to get $S' = S \setminus \{v_1\}$, a smaller set that spans W.

We ask again: *Is S' a basis for W?*

If not, then we can remove some $v_2 \in S'$ to get $S'' := S' \setminus \{v_2\}$, a smaller set that spans W.

Since $|S| < \infty$, this process will eventually terminate, and we'll be left with $\mathcal{B} := S^{(k)}$, a basis for W.

How to construct a basis from a spanning set

Example

Let $S = \{(1,0,0),\ (0,1,0),\ (1,1,0),\ (3,1,0)\} \subseteq \mathbb{R}^3$.

$W = \text{Span}(S)$ is a plane. Since $\dim(W) = 2$, a basis of W has 2 vectors.

We can remove $(1,1,0)$ and $(3,1,0)$ to get a basis $\mathcal{B} = \{(1,0,0), (0,1,0)\}$ of W.

This means that

$$\begin{aligned} W &= \{C_1(1,0,0) + C_2(0,1,0) \mid C_1, C_2, \in \mathbb{R}\} \\ &= \{(C_1, C_2, 0) \mid C_1, C_2, \in \mathbb{R}\}. \end{aligned}$$

However, note that $\{(1,0,0), (3,1,0)\}$ is also a basis for W.

This means that

$$\begin{aligned} W &= \{C_1(1,0,0) + C_2(3,1,0) \mid C_1, C_2, \in \mathbb{R}\} \\ &= \{(C_1 + 3C_2, C_2, 0) \mid C_1, C_2, \in \mathbb{R}\}. \end{aligned}$$