Lecture 1.4: Inner products and orthogonality

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Math 4340, Advanced Engineering Mathematics

Basic Euclidean geometry

Definition

Let $V = \mathbb{R}^n$. The dot product of $\mathbf{v} = (a_1, \dots, a_n)$ and $\mathbf{w} = (b_1, \dots, b_n)$ is $\mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^{n} a_i b_i$.

The length (or "norm") of $\mathbf{v} \in \mathbb{R}^n$, denoted $||\mathbf{v}||$, is the distance from $\mathbf{v}$ to $\mathbf{0}$:

$$||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{a_1^2 + \cdots + a_n^2}.$$

To understand what $\mathbf{v} \cdot \mathbf{w}$ means geometrically, we can pick a "special" $\mathbf{v}$ and $\mathbf{w}$.

- Pick $\mathbf{v}$ to be on the x-axis (i.e., $\mathbf{v} = a_1\mathbf{e}_1$).
- Pick $\mathbf{w}$ to be in the xy-plane (i.e., $\mathbf{w} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2$).

By basic trigonometry,

$$\mathbf{v} = \big(||\mathbf{v}||\,,\,0\,,\,0\,,\dots,\,0\big), \qquad \mathbf{w} = \big(||\mathbf{w}||\cos\theta\,,\,||\mathbf{w}||\sin\theta\,,\,0\,,\dots,\,0\big).$$

Proposition

The dot product of any two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ satisfies $\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}||\,||\mathbf{w}||\cos\theta$. Equivalently, the angle θ between them is

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{v}||\,||\mathbf{w}||}.$$

Basic Euclidean geometry

The following relations follow immediately:

- $(\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{v}\cdot\mathbf{v} + 2\,\mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{w} = ||\mathbf{v}+\mathbf{w}||^2$,
- $(\mathbf{v}-\mathbf{w})\cdot(\mathbf{v}-\mathbf{w}) = \mathbf{v}\cdot\mathbf{v} - 2\,\mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{w} = ||\mathbf{v}-\mathbf{w}||^2$.

Law of cosines

The last equation above says

$$||\mathbf{v}||^2 - 2\,||\mathbf{v}||\,||\mathbf{w}||\cos\theta + ||\mathbf{w}||^2 = ||\mathbf{v}-\mathbf{w}||^2,$$

which is the law of cosines.

For any unit vector $\mathbf{n}\in\mathbb{R}^n$ ($||\mathbf{n}||=1$), the projection of $\mathbf{v}\in\mathbb{R}^n$ onto $\mathbf{n}$ is $\mathrm{proj}_{\mathbf{n}}(\mathbf{v}) = \mathbf{v}\cdot\mathbf{n}$.

For example, consider $\mathbf{v}=(4,3)=4\mathbf{e}_1+3\mathbf{e}_2$ in $\mathbb{R}^2$. Note that

$$\mathrm{proj}_{\mathbf{e}_1}(\mathbf{v}) = (4,3)\cdot(1,0) = 4, \qquad \mathrm{proj}_{\mathbf{e}_2}(\mathbf{v}) = (4,3)\cdot(0,1) = 3.$$

Big idea

By defining the "dot product" in $\mathbb{R}^n$, we get for free a notion of geometry. That is, we get natural definitions of concepts such as length, angles, and projection.

To do this in other vector spaces, we need a generalized notion of "dot product."

Inner products

Definition

Let V be an $\mathbb{R}$-vector space. A function $\langle -,- \rangle\colon V \times V \to \mathbb{R}$ is a (real) inner product if it satisfies (for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V,\ c \in \mathbb{R}$):

(i) $\langle \mathbf{u}+\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$

(ii) $\langle c\mathbf{v}, \mathbf{w} \rangle = c\langle \mathbf{v}, \mathbf{w} \rangle$

(iii) $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$

(iv) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, with equaility if and only if $\mathbf{v} = 0$.

Conditions (i)–(ii) are called bilinearity, (iii) is symmetry, and (iv) is positivity.

Remark

Defining an inner product gives rise to a geometry, i.e., notions of length, angle, and projection.

- length: $||\mathbf{v}|| := \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.
- angle: $\measuredangle(\mathbf{v}, \mathbf{w}) = \theta$, where $\cos\theta = \dfrac{\langle \mathbf{v}, \mathbf{w} \rangle}{||\mathbf{v}||\ ||\mathbf{w}||}$.
- projection: if $||\mathbf{n}|| = 1$, then we can project $\mathbf{v}$ onto $\mathbf{n}$ by defining

$$\mathrm{proj}_{\mathbf{n}}(\mathbf{v}) = \langle \mathbf{v}, \mathbf{n} \rangle, \qquad \mathrm{Proj}_{\mathbf{n}}(\mathbf{v}) = \langle \mathbf{v}, \mathbf{n} \rangle \mathbf{n}.$$

This is the length or magnitude, of $\mathbf{v}$ *in the* $\mathbf{n}$*-direction.*

Orthogonality

Definition

Two vectors $\mathbf{v}, \mathbf{w} \in V$ are orthogonal if $\langle \mathbf{v}, \mathbf{w} \rangle = 0$. A set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subseteq V$ is orthonormal if $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ for all $i \neq j$ and $||\mathbf{v}_i|| = 1$ for all i.

Key idea

- Orthogonal is the abstract version of "*perpendicular*."
- Orthonormal means "*perpendicular and unit length*." An equivalent definition is

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j. \end{cases}$$

Orthonormal bases are really desirable!

Examples

1. Let $V = \mathbb{R}^n$. The standard "dot product" $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^{n} v_i w_i$ is an inner product.

   The set $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is an orthonormal basis of $\mathbb{R}^n$. We can write each $\mathbf{v} \in \mathbb{R}^n$ *uniquely* as

   $$\mathbf{v} = (a_1, \ldots, a_n) := a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n, \qquad \text{where } a_i = \text{proj}_{\mathbf{e}_i}(\mathbf{v}) = \mathbf{v} \cdot \mathbf{e}_i.$$

Orthonormal bases

Examples

2. Consider $V = \mathrm{Per}_{2\pi}(\mathbb{C})$. We can define an inner product as

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx.$$

The set

$$\left\{ e^{inx} \mid n \in \mathbb{Z} \right\} = \left\{ \dots, e^{-2ix}, e^{-ix}, 1, e^{ix}, e^{2ix}, \dots \right\}$$

is an orthonormal basis w.r.t. to this inner product.

Thus, we can write each $f(x) \in \mathrm{Per}_{2\pi}$ *uniquely* as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = c_0 + \sum_{n=1}^{\infty} c_n e^{inx} + c_{-n} e^{-inx},$$

where

$$c_n = \mathrm{proj}_{e^{inx}}(f) = \langle f, e^{inx} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx.$$

Orthonormal bases

Examples

3. Consider $V = \text{Per}_{2\pi}(\mathbb{R})$. We can define an inner product as

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x) \, dx.$$

The set

$$\left\{ \tfrac{1}{\sqrt{2}}, \ \cos x, \ \cos 2x, \dots \right\} \cup \left\{ \sin x, \ \sin 2x, \ \dots \right\}.$$

is an orthonormal basis w.r.t. to this inner product.

Thus, we can write each $f(x) \in \text{Per}_{2\pi}$ *uniquely* as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx,$$

where

$$a_n = \text{proj}_{\cos nx}(f) = \langle f, \cos nx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$b_n = \text{proj}_{\sin nx}(f) = \langle f, \sin nx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

Orthogonal bases

Important remark

Sometimes we have an orthogonal (but *not* orthonormal) basis, $\mathbf{v}_1, \dots, \mathbf{v}_n$.

There is still a simple way to decompose a vector $\mathbf{v} \in V$ into this basis.

Note that $\left\{ \frac{\mathbf{v}_1}{||\mathbf{v}_1||}, \dots, \frac{\mathbf{v}_n}{||\mathbf{v}_n||} \right\}$ is an orthonormal basis, so

$$\mathbf{v} = a_1 \frac{\mathbf{v}_1}{||\mathbf{v}_1||} + \cdots + a_n \frac{\mathbf{v}_n}{||\mathbf{v}_n||} \qquad a_i = \left\langle \mathbf{v}, \tfrac{\mathbf{v}_i}{||\mathbf{v}_i||} \right\rangle = \frac{1}{||\mathbf{v}_i||} \langle \mathbf{v}, \mathbf{v}_i \rangle = \frac{\langle \mathbf{v}, \mathbf{v}_i \rangle}{\sqrt{\langle \mathbf{v}_i, \mathbf{v}_i \rangle}}$$

$$= \frac{a_1}{||\mathbf{v}_1||} \mathbf{v}_1 + \cdots + \frac{a_n}{||\mathbf{v}_n||} \mathbf{v}_n$$

$$= c_1 \mathbf{v}_1 + \cdots + c_n \mathbf{v}_n, \qquad c_i = \frac{a_i}{||\mathbf{v}_i||} = \frac{\langle \mathbf{v}, \mathbf{v}_i \rangle}{\langle \mathbf{v}_i, \mathbf{v}_i \rangle} = \frac{\langle \mathbf{v}, \mathbf{v}_i \rangle}{||\mathbf{v}_i||^2}$$