Lecture 2.1: The fundamental theorem of linear differential equations

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Definition

A first order differential equation is of the form $y' = f(t, y)$.

A second order differential equation is of the form $y'' = f(t, y, y')$.

Linear and homogeneous ODEs

A linear 1st order ODE can be written as $y' + a(t)y = g(t)$. It is homogeneous if $g(t) = 0$.

A linear 2nd order ODE can be written as $y'' + a(t)y' + b(t)y = g(t)$. It is homogeneous if $g(t) = 0$.

Motivation for the terminology (1st order)

Consider the linear ODE $y' + a(t)y = g(t)$. Then,

$$T = \frac{d}{dt} + a(t)$$

is a linear differential operator on the space $\mathcal{C}^\infty$ of (infinitely) differentiable functions. I.e.,

$$T(y) = \left(\frac{d}{dt} + a(t)\right) y = y' + a(t)y.$$

The kernel of this operator is the set of all solutions to the "*related homogeneous ODE*", $y' + a(t)y = 0$.

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Motivation for the terminology (2nd order)

Consisder the linear ODE $y'' + a(t)y' + b(t)y = g(t)$. Then

$$T = \frac{d^2}{dt^2} + a(t)\frac{d}{dt} + b(t)$$

is a linear differential operator on the space $\mathcal{C}^\infty$ of (infinitely) differentiable functions. I.e.,

$$T(y) = \left(\frac{d^2}{dt^2} + a(t)\frac{d}{dt} + b(t)\right) y = y'' + a(t)y' + b(t)y.$$

The kernel of this operator is the set of all solutions to the "*related homogeneous ODE*", $y'' + a(t)y' + b(t)y = 0$.

Fundamental theorem of linear (homogeneous) ODEs

Let $T\colon \mathcal{C}^\infty \longrightarrow \mathcal{C}^\infty$ be a linear differential operator of order n. Then $\ker T$ is an n-dimensional subspace of $\mathcal{C}^\infty$.

What this means

- The general solution to $y' + a(t)y = 0$ has the form

$$\ker\left(\frac{d}{dt} + a(t)\right) = \Big\{C_1 y_1(t) \mid C_1 \in \mathbb{C}\Big\}.$$

  Here, $\{y_1\}$ is a basis of the "*solution space*."

- The general solution to $y'' + a(t)y' + b(t)y = 0$ has the form

$$\ker\left(\frac{d^2}{dt^2} + a(t)\frac{d}{dt} + b(t)\right) = \Big\{C_1 y_1(t) + C_2 y_2(t) \mid C_1, C_2 \in \mathbb{C}\Big\}.$$

  Here, $\{y_1, y_2\}$ is a basis of the "*solution space*."

It should be clear how this extends to ODEs of order $n > 2$.

Big idea

To solve an n^{th} order linear homogeneous ODE, we need to (somehow) find n linearly independent solutions, i.e., a basis for the solution space.

Solving linear ODEs

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Let's recall how to do this.

We'll start with **1st order ODEs**: solve $y' + a(t)y = 0$.

If you can't solve by inspection, then separate variables.

1. Solve $y' - ky = 0$.
2. Solve $y' - ty = 0$.

Remark

This always works, assuming we can evaluate $\int a(t)\,dt$.

Solving 2nd order homogeneous ODEs

Solving $y'' + a(t)y' + b(t)y = 0$ can be hard or impossible for arbitrary functions $a(t)$, $b(t)$.

One special case is when they are constants: $a(t) = p$, and $b(t) = q$.

Constant coefficients

To solve $y'' + py' + qy = 0$, guess that $y(t) = e^{rt}$ is a solution.

Example 1 (distinct real roots)

Solve $y'' + 3y' + 2y = 0$.

Solving 2nd order homogeneous ODEs

Example 2 (complex roots)

Solve $y'' - 4y' + 20y = 0$.

Summary

The general solution is a 2-dimensional vector space.

Since $y_1(t) = e^{(2+4i)t}$ and $y_2(t) = e^{(2-4i)t}$ are independent, they are a basis for the solution space.

However, the functions

$$\frac{1}{2}y_1(t) + \frac{1}{2}y_2(t) = e^{2t}\cos 4t, \qquad \frac{1}{2i}y_1(t) - \frac{1}{2i}y_2(t) = e^{2t}\sin 4t.$$

are also linearly independent solutions, and thus a different basis.

Therefore, the general solution can be expressed several different ways:

$$\text{Span}\left\{e^{(2+4i)}t,\ e^{(2-4i)t}\right\} = \text{Span}\left\{e^{2t}\cos 4t,\ e^{2t}\sin 4t\right\}.$$

We usually prefer the latter, and write an arbitrary solution as

$$y(t) = C_1e^{2t}\cos 4t + C_2e^{2t}\sin 4t = e^{2t}(C_1\cos 4t + C_2\sin 4t).$$

Solving 2nd order homogeneous ODEs

Example 3 (repeated roots)

Solve $y'' + 4y' + 4y = 0$.