Lecture 2.5: Power series solutions to differential equations

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Beyond constant coefficients

Example 1

Find the general solution to $x^2y'' + xy' - y = 0$.

Definition

A Cauchy-Euler equation is an ODE of the form

$$ax^2y'' + bxy' + cy = 0.$$

Assume there exists a solution of the form $y(x) = x^r$. Plugging this back in yields a degree-2 polynomial in r. There are three cases:

1. *Distinct real roots,* r_1, r_2: $y(x) = C_1x^{r_1} + C_2x^{r_2}$.
2. *Complex roots,* $r = a \pm bi$: $y(x) = C_1x^a\cos(b\ln x) + C_2x^a\sin(b\ln x)$.
3. *Repeated root,* r: $y(x) = C_1x^r + C_2\ln(x)x^r$.

A harder example

Example 2

Find the general solution to $y'' - 4xy' + 12y = 0$.

Example 2 (cont.)

The homogeneous ODE $y'' - 4xy' + 12y = 0$ has a power series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$, where the coefficients satisfy the following recurrence relation: $a_{n+2} = \frac{4(n-3)}{(n+2)(n+1)} a_n$.

Summary

The "power series method"

To solve $y'' - 4xy' + 12y = 0$ for $y(x)$, we took the following steps:

1. Assumed the solution has the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$.
2. Plugged the power series for $y(x)$ back into the ODE.
3. Combined into a single sum $y(x) = \sum_{n=0}^{\infty} [\quad \cdots \quad] x^n = 0$.
4. Set the x^n coefficient $[\cdots]$ equal to zero to get a recurrence $a_{n+2} = f(a_n, a_{n-1})$.

An example from physics

Legendre's equation

Find the general solution to $(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0$, where $\ell \in \mathbb{N}$ is fixed.