Lecture 2.6: Singular points and the Frobenius method

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Quick review of power series

Definitions

A power series centered at x_0 is a limit of partial sums:

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = \lim_{N\to\infty} \sum_{n=0}^{N} a_n(x-x_0)^n.$$

It converges at x if the sequence of partial sums converges. Otherwise, it diverges.

Examples

- The power series $\displaystyle\lim_{N\to\infty}\sum_{n=0}^{N}\frac{1}{n!}x^n$ converges to e^x for all $x\in(-\infty,\infty)$.
- The power series $\displaystyle\lim_{N\to\infty}\sum_{n=0}^{N}(-1)^n x^n$ converges to $\dfrac{1}{1+x}$ for all $x\in(-1,1)$. It diverges at $x=1$.

Radius of convergence

The largest number R such that if $|x-x_0|<R$, then $\displaystyle\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges.

Ordinary vs. singular points of ODEs

Definitions

A function $f(x)$ is real analytic at x_0 if $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ for some $R > 0$.

Definition

Consider the ODE $y'' + P(x)y' + Q(x)y = 0$.

- The point x_0 is an ordinary point if $P(x)$ and $Q(x)$ are real analytic at x_0.
- Otherwise x_0 is a singular point, which is:
  - regular if $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ are real analytic.
  - irregular otherwise.

Examples (at $x_0 = 0$)

- *Ordinary*: $y'' + xy + y = 0$.
- *Regular singular*: $x^2y'' + xy' + y = 0$.
- *Irregular singular*: $x^2y'' + y' + y = 0$.

When does an ODE have a power series solution?

Theorem of Frobenius

Consider an ODE $y'' + P(x)y' + Q(x)y = f(x)$. If x_0 is an ordinary point, and $P(x)$, $Q(x)$, and $f(x)$ have radii of convergence R_P, R_Q, and R_f, respectively, then there is a power series solution

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \qquad R = \min\{R_P, R_Q, R_f\}.$$

If x_0 is a regular singular point and $(x - x_0)P(x)$, $(x - x_0)^2 Q(x)$, and $f(x)$ have radii of convergence R_P, R_Q, and R_f, respectively, then there is a generalized power series solution

$$y(x) = (x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0)^n, \qquad R = \min\{R_P, R_Q, R_f\},$$

for some constant r.

Example

Find the general solution to $2xy'' + y' + y = 0$.

Applications of the Frobenius method

Examples from physics and engineering

- Cauchy-Euler equation: $x^2y'' + axy' + by = 0$. *Arises when solving Laplace's equation in polar coordinates.*
- Legendre's equation: $(1-x^2)y'' - 2xy' + n(n+1)y = 0$. *Used for modeling spherically symmetric potentials in the theory of Newtonian gravitation and in electricity & magnetism (e.g., the wave equation for an electron in a hydrogen atom).*
- Chebyshev's equation: $(1-x^2)y'' - xy' + n^2y = 0$. *Arises in numerical analysis techniques.*
- Hermite's equation: $y'' - 2xy' + 2ny = 0$. *Used for modeling simple harmonic oscillators in quantum mechanics.*
- Bessel's equation: $x^2y'' + xy' + (x^2 - n^2)y = 0$. *Used for analyzing vibrations of a circular drum.*
- Laguerre's equation: $xy'' + (1-x)y' + ny = 0$. *Arises in a number of equations from quantum mechanics.*
- Airy's equation: $y'' - k^2xy = 0$. *Models the refraction of light.*