Lecture 2.7: Bessel's equation

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Bessel's equation

The following ODE will arise when we solve the wave equation in polar coordinates:

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \qquad \nu \in \mathbb{Z}_{\geq 0}.$$

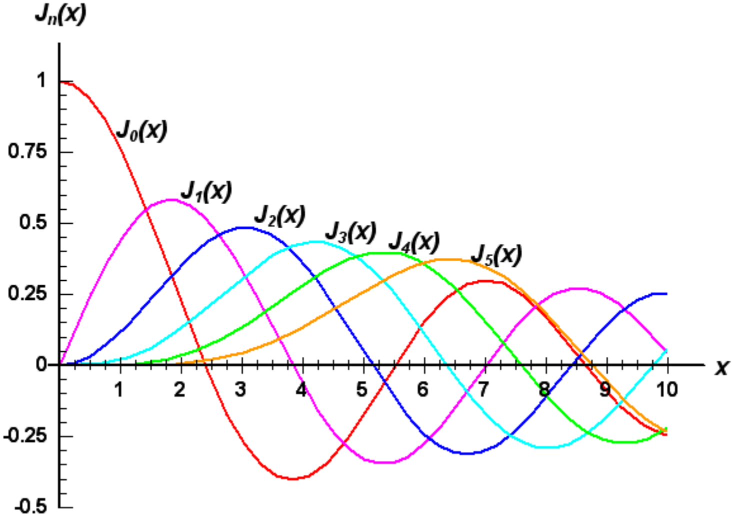
Bessel's equation: $x^2y'' + xy' + (x^2 - \nu^2)y = 0$

We assumed a generalized power series solution $y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$, $a_0 \neq 0$, and derived

$$(r^2 - \nu^2)a_0 = 0, \qquad [(r+1)^2 - \nu^2]a_1 = 0, \qquad [(n+r)^2 - \nu^2]a_n + a_{n-2} = 0, \text{ for } n \geq 2.$$

Bessel functions of the first kind

$$J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(\nu+m)!} \left(\frac{x}{2}\right)^{2m+\nu}.$$



Summary so far

We solved Bessel's equation: $x^2y'' + xy' + (x^2 - \nu^2)y = 0$, using the Frobenius method, and found two generalized power series solutions:

$$y_1(x) = x^{\nu} \sum_{n=0}^{\infty} a_n x^n, \qquad y_2(x) = x^{-\nu} \sum_{n=0}^{\infty} a_n x^n.$$

Unfortuntely, if $\nu \in \mathbb{Z}$, these are *not* linearly independent.

Since the Wronskian is $W(y_1, y_2) = e^{-\int \frac{1}{x}} = \frac{c}{x}$, both solutions can't be bounded as $x \to 0$.

We called this first solution a Bessel function of the first kind. For each fixed ν, it is

$$J_{\nu}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(\nu+m)!} \left(\frac{x}{2}\right)^{2m+\nu}.$$

To find a second solution, we need to use variation of parameters: assume

$$y_2(x) = v(x) J_{\nu}(x),$$

and solve for $v(x)$. Once normalized, this solution $Y_{\nu}(x)$ is called a Bessel function of the second kind, and satisfies

$$Y_{\nu}(x) = \lim_{\alpha \to \nu} \frac{J_{\alpha}(x)\cos(\alpha\pi) - J_{-\alpha}(x)}{\sin(\alpha\pi)}.$$

Bessel functions of the second kind

$$J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(\nu+m)!}\left(\frac{x}{2}\right)^{2m+\nu}, \qquad Y_\nu(x) = \lim_{\alpha\to\nu} \frac{J_\alpha(x)\cos(\alpha\pi) - J_{-\alpha}(x)}{\sin(\alpha\pi)}.$$