Lecture 3.4: Fourier sine and cosine series

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Math 4340, Advanced Engineering Mathematics

Motivation

When we study partial differential equations (PDEs), we'll see problems like this.

Example 1

Consider a metal bar of length $L = 1$, insulated along its interior but not its endpoints, sitting in a 0° room. Initally, the bar is 100°. Find the function $u(x,t)$ that describes the temperature of the bar.

The function $u(x,t)$ must satisfy the following PDE called the heat equation:

$$\underbrace{\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}}_{\text{heat equation}}, \qquad \underbrace{u(0,t) = u(1,t) = 0}_{\text{boundary conditions}}, \qquad \underbrace{u(x,0) = 100}_{\text{initial condition}}.$$

As we'll see, the "general solution" to the PDE subject to these boundary condtions is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)\, e^{-(cn\pi)^2 t}.$$

The last step is to plug in $t = 0$ and solve for the coefficients b_n:

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) = 100, \qquad \text{for } 0 < x < 1.$$

To do this, we must express the function $f(x) = 100$, on $0 < x < 1$ as a Fourier sine series!

A Fourier sine series

Example 1

Express the function $f(x) = 100$ as $f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$ for $0 < x < 1$.

Even and odd extensions

Definition

Let $f(x)$ have domain $(0, L)$. There are several natural ways to make $f(x)$ periodic:

- the *periodic extension* of $f(x)$,
- the even extension of $f(x)$,
- the odd extension of $f(x)$.

Fourier sine and cosine series

Definition

Let $f(x)$ be a function defined for $0 < x < L$.

- The Fourier cosine series of f is the Fourier series of the even extension of f.
- The Fourier sine series of f is the Fourier series of the odd extension of f.

Computations

Example 2

Let $f(x) = x$ for $0 < x < 1$. Compute the Fourier sine and cosine series of $f(x)$.