Lecture 3.6: Real vs. complex Fourier series

Matthew Macauley

Department of Mathematical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

Math 4340, Advanced Engineering Mathematics

Overview

Last time, we derived formulas for the complex Fourier series of a function.

Complex Fourier series

If $f(x)$ is a piecewise continuous $2L$-periodic function, then we can write

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{i\pi nx}{L}} = c_0 + \sum_{n=1}^{\infty} \left(c_n e^{\frac{i\pi nx}{L}} + c_{-n} e^{-\frac{i\pi nx}{L}}\right)$$

where

$$c_0 = \langle f, 1 \rangle = \frac{1}{2L} \int_{-L}^{L} f(x)\,dx, \qquad c_n = \langle f, e^{\frac{i\pi nx}{L}} \rangle = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-\frac{i\pi nx}{L}}\,dx.$$

Here, we will see how to go between the real and complex versions of a Fourier series.

It's just a simple application of the following identities that we've already seen:

Euler's formula (and consequences)

- $e^{i\theta} = \cos\theta + i\sin\theta, \qquad e^{-i\theta} = \cos\theta - i\sin\theta,$
- $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}.$

From the real to the complex Fourier series

Proposition

The complex Fourier coefficients of $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}$ are

$$c_n = \frac{a_n - ib_n}{2}, \qquad c_{-n} = \frac{a_n + ib_n}{2}.$$

From the complex to the real Fourier series

Proposition

The real Fourier coefficients of $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{i\pi nx}{L}}$ are

$$a_n = c_n + c_{-n}, \qquad b_n = i(c_n - c_{-n}).$$

Computations

Example 1: square wave

Find the complex Fourier series of $f(x) = \begin{cases} 1, & 0 < x < \pi \\ -1, & \pi < x < 2\pi. \end{cases}$

Computations

Example 2

Compute the real Fourier series of the 2π-periodic extension of the function e^x defined on $-\pi < 0 < \pi$.