Lecture 5.1: Fourier's law and the diffusion equation

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Math 4340, Advanced Engineering Mathematics

Partial differential equations

Definition

Let $u(x, t)$ be a 2-variable function. A partial differential equation (PDE) is an equation involving u, x, t, and the partial derivatives of u.

PDEs vs. ODEs

ODEs have a unifying theory of existence and uniqueness of solutions.

PDEs have no such theory.

PDEs arise from physical phenomena and modeling.

Motivation

The diffusion equation is a PDE that can model the motion of a number of physical processes such as:

- smoke in the air,
- dye in a solution,
- heat through a medium.

Let $u(x, y, z, t)$ be the concentration (or temperature, etc.) at position (x, y, z) and time t.

Let $\mathbf{F}$ be the vector field that describes the flow of smoke (or heat, etc.)

Goal. Relate how u varies with respect to time to how it varies in space.

Definition

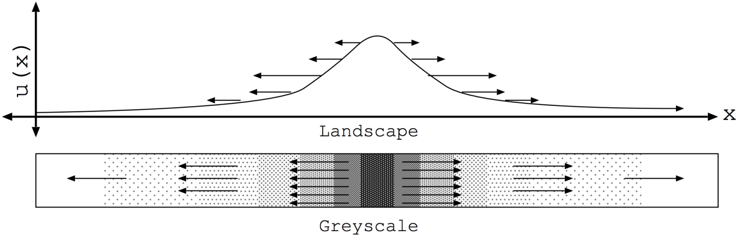
The diffusion equation (or heat equation) is the PDE

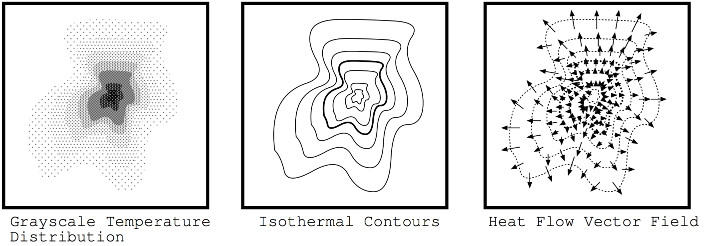
$$\frac{\partial u}{\partial t} = k \underbrace{\nabla^2 u}_{\text{Laplacian}} = k \underbrace{\nabla \cdot \nabla u}_{\text{div}(\nabla u)} = k \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = k(u_{xx} + u_{yy} + u_{zz}).$$

Fourier's law of diffusion

Material flows from regions of greater to lesser concentration, at a rate propotional to the gradient:

$$\mathbf{F} = -k\nabla u.$$





Derivation

Steps to deriving the diffusion equation

1. Fourier's law: $\mathbf{F} = -k\nabla u$.
2. Relate $\mathbf{F}$ and $\frac{\partial u}{\partial t}$ by the divergence theorem:

$$\iiint_D \operatorname{div} \mathbf{F} \, dV = \text{“Flux through } S\text{''} = \oiint (\mathbf{F} \cdot \mathbf{n}) \, dS.$$

By the divergence theorem,

$$\iiint_D \operatorname{div} \mathbf{F} \, dV = -\frac{\partial}{\partial t} \iiint_D u \, dV = -\iiint_D \frac{\partial u}{\partial t} \, dV.$$

holds for *any* region D. Thus,

$$\operatorname{div} \mathbf{F} = -\frac{\partial u}{\partial t}.$$

Now plug this into $\mathbf{F} = -k\nabla u$:

$$-\frac{\partial u}{\partial t} = \operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \nabla \cdot (-k\nabla u) \qquad \Longrightarrow \qquad \frac{\partial u}{\partial t} = k\nabla^2 u.$$

In one-dimension, this reduces to $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$, or just $u_t = ku_{xx}$.

Diffusion in one dimension (non-uniform)

Consider a pipe of length L containing a medium. The diffusion equation is the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(D(u,x)\frac{\partial u}{\partial x}\Big), \qquad \text{where}$$

$$\begin{aligned} u(x,t) &= \text{ density of the diffusing material at position } x \text{ and time } t \\ D(u,x) &= \text{ collective diffusion coefficient for density } u \text{ and position } x. \end{aligned}$$

Assuming that the diffusion coefficient is constant, the diffusion equation becomes

$$u_t = c^2 u_{xx}, \qquad\qquad c^2 = -D.$$

Heat flow in one dimension (non-uniform)

Consider a bar of length L that is insulated along its interior. The heat equation is the PDE

$$\rho(x)\sigma(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(\kappa(x)\frac{\partial u}{\partial x}\Big), \qquad \text{where}$$

$$\begin{aligned} u(x,t) &= \text{ temperature of the bar at position } x \text{ and time } t \\ \rho(x) &= \text{ density of the bar at position } x \\ \sigma(x) &= \text{ specific heat at position } x \\ \kappa(x) &= \text{ thermal conductivity at position } x. \end{aligned}$$

Assuming that the bar is "uniform" (i.e., ρ, σ, and κ are constant), the heat equation is

$$u_t = c^2 u_{xx}, \qquad\qquad c^2 = \kappa/(\rho\sigma).$$

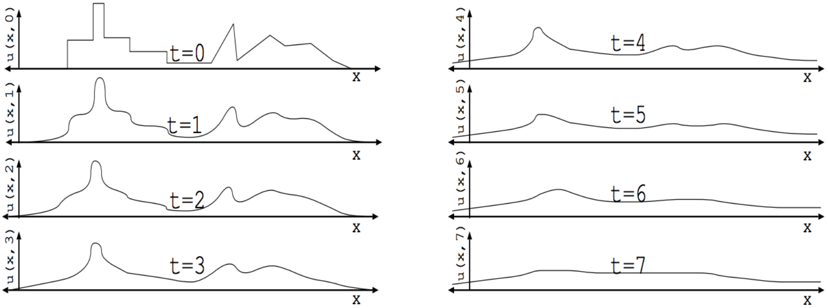
Adding boundary and initial conditions

Example 1a

The following is a boundary / initial value problem (B/IVP) for the heat equation in one dimension:

$$\underbrace{u_t = c^2 u_{xx}}_{\text{heat equation (PDE)}}\ , \qquad \underbrace{u(0,t) = u(L,t) = 0}_{\text{boundary conditions}} \qquad \underbrace{u(x,0) = x(L-x)}_{\text{initial condition}}\ .$$

The following is a picture of what a solution looks like over time.



Solving PDEs

PDEs, like ODEs, can be homogeneous or inhomogeneous. Like ODEs, we'll solve them by:

1. Solving the related homogeneous equation
2. Finding a particular solution (almost always a "steady-state" solution)
3. Adding these two solutions together.

Most common homogeneous PDEs can be solved by a method called separation of variables.

Separation of variables (in one dimension)

How to solve a PDE like

$$\underbrace{u_t = c^2 u_{xx}}_{\text{heat equation (PDE)}}, \qquad \underbrace{u(0,t) = u(L,t) = 0}_{\text{boundary conditions}} \qquad \underbrace{u(x,0) = x(L-x)}_{\text{initial condition}}.$$

1. Assume that there is a solution of the form $u(x,t) = f(x)g(t)$.
2. Plug this back into the PDE and solve for $f(x)$ and $g(t)$. (Separate variables!)
3. You'll get a BVP for $f(x)$ and an ODE for $g(t)$.
4. Solve these ODEs. You'll get a solution $u_n(x,t)$ for each $n = 0, 1, 2, \dots$.
5. By superposition, the general solution is $u(x,t) = \sum_{n=0}^{\infty} c_n\, u_n(x,t)$.
6. Use the initial condition to find the c_n's.

Solving the heat equation

Example 1a

Recall the following is a boundary / initial value problem (B/IVP) for the heat equation in one dimension:

$$\underbrace{u_t = c^2 u_{xx}}_{\text{heat equation (PDE)}}, \qquad \underbrace{u(0,t) = u(L,t) = 0}_{\text{boundary conditions}} \qquad \underbrace{u(x,0) = x(L-x)}_{\text{initial condition}}.$$

Solving the heat equation

Example 1a (cont.)

The general solution to the BVP for the heat equation

$$\underbrace{u_t = c^2 u_{xx}}_{\text{heat equation (PDE)}}, \qquad \underbrace{u(0,t) = u(L,t) = 0}_{\text{boundary conditions}} \qquad \underbrace{u(x,0) = x(L-x)}_{\text{initial condition}}.$$

is $u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-(cn\pi/L)^2 t}$. Finally, we'll use the initial condition.

Solving the heat equation

Example 1a (cont.)

The particular solution to the heat equation that satisfies the following boundary and initial conditions

$$\underbrace{u_t = c^2 u_{xx}}_{\text{heat equation (PDE)}}, \qquad \underbrace{u(0,t) = u(L,t) = 0}_{\text{boundary conditions}} \qquad \underbrace{u(x,0) = x(L-x)}_{\text{initial condition}}$$

is $u(x,t) = \sum_{n=1}^{\infty} 4\left(\frac{L}{n\pi}\right)^3 [1-(-1)^n] \sin\left(\frac{n\pi x}{L}\right) e^{-(cn\pi/L)^2 t}$.